MARKOV PROCESSES WITH IDENTICAL HITTING DISTRIBUTIONS

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1. **Introduction.** Throughout X and X^* are to be time homogeneous Markov processes taking values in a locally compact, noncompact, separable metric space E, and both satisfying Hunt's condition (A) [**2**, pp. 48–50]. The purpose of this note is to give rather general conditions under which there exists a continuous random time change $\tau(t)$, in the sense of [**4**, p. 104], such that $X(\tau(t))$ and $X^*(t)$ are equivalent, that is that they have the same transition function. Obviously a necessary condition, at least if $\tau(t)\to\infty$ as $t\to\infty$, is that the two processes have the same hitting distributions in the sense of hypothesis (h_1) below. Our theorem is that under a mild additional assumption this condition is also sufficient. A full proof will be published elsewhere.

2. **Hypotheses.** Let $P(t, x, A)$ be the transition function for the process X, P_x and E_x the probabilities and expectations for X starting at x, T_A the infimum of the strictly positive t such that $X(t)$ is in the subset A of E and $H_A(x, B)=P_x(X(T_A)\in B,\ T_A<\infty)$, A and B being Borel sets. Analogous quantities for X^* are denoted by P^*, E^*, T^* and H^* with appropriate arguments. Our hypotheses are these: (h_1) for each x in E and compact K, $H_K(x, \cdot)=H_K^*(x, \cdot)$, and ($h_2$) there is an increasing sequence $\{G_n\}$ of compact sets whose union is E and such that, for each x and n, $P_x(T_{G_n^c}<\infty)=1$. The c here denotes complement.

3. **Outline of construction.** Fix one of the sets $G=G_n$ and suppress the subscript. If $f_\lambda(x)=E_x^*(1-\exp(-\lambda T_{G^c}^*))$, $\lambda>0$, then f_λ is excessive for the process X^* terminated when it first leaves G. By a theorem of Dynkin [**1**] it is then also excessive for X similarly terminated. One can show that f_λ is regular enough that arguments of Šur [**5**] and Volkonskii [**6**] apply to it and yield a continuous additive functional $\phi_\lambda(t)$ satisfying $E_x\phi_\lambda(T_{G^c})=f_\lambda(x)$. One next shows that $\lambda^{-1}\phi_\lambda(t)$ increases, as $\lambda\to 0$, to a continuous strictly increasing additive functional which, reintroducing the index n, we call $\phi^n(t)$. The ϕ^n for varying n are shown to be compatible in the sense that if $m>n$ then for

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all x with P_x probability one $\phi^n(t) = \phi^m(t)$ throughout the interval $t < T_{G_n^c}$. The limit as $n \to \infty$ of $\phi^n(t)$ is a continuous additive functional $\phi(t)$.

The desired time change $\tau(t)$ is the functional inverse to ϕ. That $X(\tau(t))$ is equivalent to $X^*(t)$ follows from the computation of certain potentials.

4. Remarks. Usually the hypothesis (h_2) may be eliminated. For example if the semi-group for one of the processes leaves invariant the space of bounded continuous functions on E then (h_1) alone implies the existence of the desired time change.

In [3] there appears a more explicit form of our result in case X is Brownian motion in Euclidean space and X^* is a diffusion process with the same hitting distributions. The construction makes use of potential theoretic facts which are available for transition functions having a sort of symmetry, but not for those as general as the ones we consider here.

The results announced here are also valid for processes having finite terminal times.

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