MEROMORPHIC MINIMAL SURFACES

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1. **Introduction.** The Nevanlinna theory [4] of meromorphic functions of a complex variable is based primarily on Jensen's formula [3]. In this note we shall present a generalization of this formula for minimal surfaces and shall briefly indicate how the formula thus generalized can be applied to yield an extension of the theory to include these surfaces.

2. **Minimal surfaces.** By a theorem of Weierstrass [6], a necessary and sufficient condition that a surface

$$S: x_j = x_j(u, v), \qquad j = 1, 2, 3, \tag{1}$$

given in terms of isothermic parameters, that is, parameters u, v such that

$$E = G = \lambda(u, v), \qquad F = 0, \tag{2}$$

be minimal is that the coordinate functions be harmonic. Such a set of harmonic functions is said to be a *triple of conjugate harmonic functions* [2], in analogy with the couples of conjugate harmonic functions that form analytic functions of a complex variable.

If the functions $x_j(u, v)$ are harmonic in a deleted circular neighborhood N of a point P, then they can be represented [5] in N by series of the form

$$x_j = \sum_{k=-\infty}^{\infty} r^k(a_{j,k} \cos k\theta + b_{j,k} \sin k\theta) + c_j \log r, \tag{3}$$

where (r, θ) are polar coordinates with pole at P.

An investigation of the relations (2) shows that the triple of conjugate harmonic functions (3) cannot have a logarithmic singularity; that is, we cannot have $a_{j,k} = b_{j,k} = 0$, $j = 1, 2, 3$, for all $k < 0$, but not all $c_j = 0$. To be sure, the relations $x_1 = \Re(\log w - \frac{1}{2}w^2)$, $x_2 = \Re i(\log w + \frac{1}{2}w^2)$, $x_3 = \Re 2w$, where $w = u + iv$, give a minimal surface in isothermic representation; but the second of these relations is not a (single-valued) function.

If $a_{j,k} = b_{j,k} = 0$, $j = 1, 2, 3$, for all $k < t$, but some $a_{j,t}$ or $b_{j,t} \neq 0$, then [1] we have

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$$\left\{\sum_{j=1}^{3}[x_j(u,v)]^2\right\}^{1/2} = r^t\left[\left(\sum_{j=1}^{3} a_{j,t}^2\right)^{1/2} + o(1)\right].$$

For $t>0$, we say that S has a *zero* of order t at the point P, while for $t<0$ we say that S has a *pole* of order $|t|$ at P. Clearly, the zeros and poles of S must be isolated.

If, except for poles, the functions (1) are a triple of conjugate harmonic functions throughout a finite domain D, then we shall say that S is a *meromorphic minimal surface* for (u, v) in D, and shall say that S is given in *typical representation* by the functions (1). If the domain D is the entire finite plane, then we shall say that the minimal surface is *entire*.

For example, the functions

$$x_1 = \Re\left(\frac{1}{w} + w\right), \qquad x_2 = \Re i\left(\frac{1}{w} - w\right), \qquad x_3 = \Re 2\log w$$

give a typical representation of an entire meromorphic minimal surface having a pole of order 1 at the origin. Thus the coordinate functions of a minimal surface in typical representation can contain logarithmic terms, whereas in complex-variable theory a function given by an expression of the form (3) has a (single-valued) harmonic conjugate if and only if $c_j=0$.

The Laplacian of the logarithm of the distance function [2] for a meromorphic minimal surface is given by

$$\Delta \log\left(\sum_{j=1}^{3} x_j^2\right)^{1/2} = \frac{2\lambda\left(\sum_{j=1}^{3} x_j\nu_j\right)^2}{\left(\sum_{j=1}^{3} x_j^2\right)^2}, \tag{4}$$

where the ν_j are the direction cosines of the normal to the surface. It can be shown that (4) remains continuous even at the zeros and poles of the surface.

3. Jensen's formula. Let the functions $x_j(w)$, $w=u+iv$, represent a meromorphic minimal surface S in a domain D containing the circular disc $|w| \leqq R$ in its interior. For the representation (3) of the x_j in the neighborhood of the origin, suppose that $a_{j,k}=b_{j,k}=0$, $j=1, 2, 3$, for all $k<t$, but some $a_{j,t}$ or $b_{j,t}\neq 0$. In $0<|w|<R$, let the zeros and poles of S be at $a_1, a_2, \cdots, a_p$ and $b_1, b_2, \cdots, b_q$, respectively, a zero or pole of multiplicity s being counted s times, and suppose that S has no zero or pole on $|w|=R$.

For any circular disc $|w| \leqq r$, $r<R$, Green's formula $\int(dg/dn)ds$

$=\iint \Delta g dA$, applied to the function

$$g(w) = \log \left| \left\{ \sum_{j=1}^{3} [x_j(w)]^2 \right\}^{1/2} \frac{R^t}{w^t} \prod_{p=1}^{m} \frac{R^2 - \bar{a}_p w}{R(w - a_p)} \prod_{q=1}^{n} \frac{R(w - b_q)}{R^2 - \bar{b}_q w} \right|,$$

yields an equation that, when divided by r and then integrated with respect to r from $r=0$ to $r=R$, by (4) gives

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left\{ \sum_{j=1}^{3} [x_j(\mathrm{R}e^{i\theta})]^2 \right\}^{1/2} d\theta$$

$$(5) \quad = \log \left| \left(\sum_{j=1}^{3} a_{j,t}^2 \right)^{1/2} R^{t+m-n} \frac{\prod_{q=1}^{n} b_q}{\prod_{p=1}^{m} a_p} \right|$$

$$+ \int_0^R \left\{ \frac{1}{r} \int_0^r \int_0^{2\pi} \frac{\left\{ \sum_{j=1}^{3} [x_j(\rho e^{i\theta})][\nu_j(\rho e^{i\theta})] \right\}^2}{\pi \left\{ \sum_{j=1}^{3} [x_j(\rho e^{i\theta})]^2 \right\}^2} \lambda(\rho e^{i\theta}) \rho d\rho d\theta \right\} dr.$$

This generalizes the formula of Jensen.

4. **The Nevanlinna theory.** We let $a=(a_1, a_2, a_3)$ denote an arbitrary point in three-dimensional Euclidean space, to which we adjoin a single ideal *point at* ∞. We let $n(r, a; S)$, $a \neq \infty$, and $n(r, \infty; S)$ denote the number of zeros of the surface S_a: $x_j = x_j(w) - a_j$ and the number of poles of S, respectively, in $|w| \leqq r$, and adjoin the function

$$h(r, a; S) = \int_0^r \int_0^{2\pi} \frac{\left\{ \sum_{j=1}^{3} [x_j(\rho e^{i\theta}) - a_j][\nu_j(\rho e^{i\theta})] \right\}^2}{\pi \left\{ \sum_{j=1}^{3} [x_j(\rho e^{i\theta}) - a_j]^2 \right\}^2} \lambda(\rho e^{i\theta}) \rho d\rho d\theta, \qquad a \neq \infty,$$

$$h(r, \infty; S) = 0.$$

For $a \neq \infty$, this last function gives a measure, relative to the circular-disc area $\pi \sum_{j=1}^{3} (x_j - a_j)^2$, of the surface area of S weighted by a measure, namely $\cos^2(x-a, \nu)$, of the deviation of its normal from perpendicularity to the radius vector $x-a$.

To the enumerative function

$$N(R, a; S) = \int_0^R \frac{n(r, a; S) - n(0, a; S)}{r} dr + n(0, a; S) \log R$$

and the proximity function

$$m(R, a; S) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left\{\sum_{j=1}^{3} [x_j(Re^{i\theta}) - a_j]^2\right\}^{-1/2} d\theta, \qquad a \neq \infty,$$

$$m(R, \infty; S) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left[\sum_{j=1}^{3} x_j^2(Re^{i\theta})\right]^{1/2} d\theta,$$

we adjoin the *skewness function*

$$H(R, a; S) = \int_0^R \frac{h(r, a; S)}{r}\, dr.$$

The sum

$$A(R, a; S) = N(R, a; S) + m(R, a; S) + H(R, a; S)$$

might be called the *affinity* of S for the value a. In particular,

$$A(R, \infty; S) = N(R, \infty; S) + m(R, \infty; S) = T(R; S)$$

might be called the *characteristic function* of S.

Now the extension (5) of Jensen's formula can be written as

$$\begin{aligned} A(R, 0; S) &= A(R, \infty; S) - \log\left(\sum_{j=1}^{3} a_{j,t}^2\right)^{1/2} \\ &= T(R; S) - \log\left(\sum_{j=1}^{3} a_{j,t}^2\right)^{1/2}. \end{aligned} \tag{6}$$

This admits a further extension to the *spherical characteristic*, leads directly to a generalization of Nevanlinna's first fundamental theorem, and so on.

It might be noted that (6) applies in particular to plane maps, e.g., to $x_1 = u$, $x_2 = v$, $x_3 = 5$, and thus yields a measure of the affinity of a meromorphic function $f(w)$ of a complex variable to *all* points of space, not merely to points of the complex plane.

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