sional Brownian motion, and Kac [2] for a restricted class of random walks.

References

1. G. Hunt, *Some theorems on Brownian motion*, Trans. Amer. Math. Soc. **81** (1956), 294–319.

2. M. Kac, *A class of limit theorems*, Trans. Amer. Math. Soc. **84** (1957), 459–471.

3. F. Spitzer, *Hitting probabilities*, J. Math. Mech. **11** (1962), 593–614.

Cornell University and
Stanford University

CLASSIFICATION OF AEA FORMULAS BY LETTER ATOMS

BY BURTON DREBEN, A. S. KAHR, AND HAO WANG

Communicated by A. M. Gleason, May 11, 1962

The class G of all closed prenex schemata in the form $AxEuAyMxuy$ whose quantifier-free matrices $Mxuy$ contain only dyadic schematic letters was shown in [2] to be a reduction class for quantification theory. Here we shall study the decision problems of various subclasses of this unsolvable class. Since a dyadic schematic letter may be followed by xx, xu, ux, uu, yy, xy, yx, uy, or yu, any letter atom occurring in a given matrix $Mxuy$ is in one of these nine letter atomic forms. The subclasses of G to be studied will be specified in terms of these forms.

Consider the four letter atomic forms xy, yx, uy, yu. First take any three of them. From [2] we know that any subclass of G which includes all schemata whose letter atoms are in just these three forms is a reduction class and hence is unsolvable. Now take any two of the four forms. Combining them with the other five forms yields a subclass of G. In this way we obtain six subclasses of G which divide into three pairs: $J=\{xy, uy\}$, $J^*=\{yx, yu\}$, $L=\{xy, yx\}$, $L^*=\{uy, yu\}$, $Q=\{xy, yu\}$, $Q^*=\{yx, uy\}$. We discuss these three pairs of classes in turn.

The subclasses J and J^* are solvable and contain axioms of infinity, i.e., contain some schemata having just infinite models. This is significant because thus far no naturally specifiable infinite class of schemata containing axioms of infinity has been shown solvable. Since the arguments for J and J^* are analogous, we discuss only J.

Let S be any member of J and let S contain N dyadic schematic letters $F_1, \cdots, F_N$. Without loss of generality, we can assume that S contains exactly $7N$ distinct letter atoms (N letter atoms in each of the 7 permissible forms) and is in perfectly developed disjunctive normal form. Hence the matrix $Mxuy$ of S is a truth-functional disjunction of δ distinct letter disjuncts each of which is a conjunction of $7N$ signed occurrences of distinct letter atoms, where $1 \leqq \delta \leqq 2^{7N}$. By the Skolem-Herbrand-Gödel Theorem, S has a model if and only if no finite conjunction of numerical instances $M(i, i+1, j)$ is truth-functionally inconsistent for all $i, j \geqq 0$. But $Mxuy$ is a disjunction of δ distinct letter disjuncts. So each numerical instance $M(i, i+1, j)$ is a disjunction of δ distinct *number disjuncts*; that is, the numerical instance $M(i, i+1, j)$ has the form

$$D_1(i, i+1, j) \vee D_2(i, i+1, j) \vee \cdots \vee D_\delta(i, i+1, j).$$

Hence the schema S has a model if and only if for each finite set of numerical instances of $Mxuy$ there is a truth-functionally consistent way of selecting simultaneously one number disjunct from each of the numerical instances in the set. Assume now that such a consistent *Herbrand selection* H can be made from the finite set of numerical instances $\{M(0, 1, 0), \cdots, M(\epsilon, \epsilon+1, \epsilon)\}$ where $\epsilon = 2^\delta$. On this assumption, we show how to select simultaneously a number disjunct from each of the infinity of numerical instances of $Mxuy$ such that each finite set of selections is consistent. Thus the schema S has a model if and only if the set $\{M(0, 1, 0), \cdots, M(\epsilon, \epsilon+1, \epsilon)\}$ is truth-functionally consistent.

For each fixed number $c \leqq \epsilon$ and for all $j \leqq \epsilon$ consider the finite set of all number disjuncts in the Herbrand selection H of the form $D_d(c, c+1, j)$ where $1 \leqq d \leqq \delta$. Call the set of all the distinct subscripts d of such number disjuncts the *index of c over H*. Since there are at most $2^\delta - 1$ nonempty subsets of δ distinct subscripts and since $\epsilon = 2^\delta$, by the pigeon hole principle there must be two distinct numbers a and b each less than $\epsilon - 1$ and each having an index over H identical with the index of the other over H. Let $a < b$. Define now for all $n \leqq \epsilon$ two functions α^1 and β^1 thus:

(1) if $n = b$, then $\alpha^1(n) = a$;

(2) if $n \neq b$, then $\alpha^1(n)$ is the least number $\leqq \epsilon$ such that the number disjuncts $D_d(b, b+1, n)$ and $D_{d'}(a, a+1, \alpha^1(n))$ occur in H and are cosubscripted, i.e., $d = d'$.

(3) if $n = a+1$, then $\beta^1(n) = b+1$.

(4) if $n \neq a+1$, then $\beta^1(n)$ is the least number $\leqq \epsilon$ such that the

number disjuncts $D_d(a, a+1, n)$ and $D_{d'}(b, b+1, \beta^1(n))$ occur in H and are cosubscripted.

Also, let $\alpha^0(n)=\beta^0(n)=n$; $\alpha^{m+1}(n)=\alpha(\alpha^m(n))$, and $\beta^{m+1}=\beta(\beta^m(n))$, for all $m\geqq 0$.

Hence for each $n\leqq\epsilon$ and each $m\geqq 0$, (Γ) the *number atoms* $F_f(b+1, \alpha^m(n))$ and $F_f(a+1, \alpha^{m+1}(n))$ are cosigned in H and (γ) the number atoms $F_f(a, \beta^m(n))$ and $F_f(b, \beta^{m+1}(n))$ are cosigned in H. Moreover, (Δ) for each $n\leqq\epsilon$ and for all $m\geqq 0$ the number atoms $F_f(n, n)$, $F_f(\alpha^m(n), \alpha^m(n))$, and $F_f(\beta^m(n), \beta^m(n))$ are cosigned in H $(f=1, \cdots, N)$. In particular, the number atoms $F_f(a, a+1)$ and $F_f(b, b+1)$ are cosigned in H, the number atoms $F_f(a+1, a)$ and $F_f(b+1, b)$ are cosigned in H, and the number atoms $F_f(a, a)$ and $F_f(b, b)$ are cosigned in H.

We now define a mapping μ on the set of all numerical instances of S and into the set of instances $\{M(0, 1, 0), \cdots, M(\epsilon, \epsilon+1, \epsilon)\}$. We then complete the proof of the solvability of the class J by showing consistent the infinite Herbrand selection H_ω formed by selecting from each numerical instance $M(i, i+1, j)$ the unique number disjunct cosubscripted with the number disjunct contributed by $\mu(M(i, i+1, j))$ to the finite Herbrand selection H.

The definition of the mapping μ is in three cases:

CASE I. Let $i, j\leqq b$. Then set $\mu(M(i, i+1, j))=M(i, i+1, j)$.

CASE II. Let $i>b$, $i\geqq j$, $j\geqq 0$, and $b=a+s$. Then $i=b+ks+q$, where $0\leqq k$ and $1\leqq q\leqq s$.

(1) If $i=j$, then set $\mu(M(i, i+1, j))=M(a+q, a+q+1, a+q)$.

(2) If $i>j$, then set

$$\mu(M(i, i+1, j)) = M(a+q, a+q+1, \alpha^{k-g+1}(j-gs))$$

where g is the least number such that $j-gs<b$ $(0\leqq g\leqq k+1)$.

CASE III. Let $j>b$, $j>i$, $i\geqq 0$, and $b=a+s$. Then $j=b+ks+q$, where $0\leqq k$ and $1\leqq q\leqq s$.

(1) If $j=i+1$, then set $\mu(M(i, i+1, j))=M(a+q-1, a+q, a+q)$.

(2) If $j>i+1$, then set

$$\mu(M(i, i+1, j)) = M(i-gs, i-gs+1, \beta^{k-g+1}(a+q))$$

where g is the least number such that $i-gs<b$ $(0\leqq g\leqq k+1)$.

To show the consistency of H_ω we have to show for each number atom that all of its occurrences in H_ω are cosigned. So let $F_f(i, j)$ be any number atom $(i, j\geqq 0)$.

(1) Let $i=j$. If $i=j<b$, then all occurrences of $F_f(i, j)$ in H_ω have the same sign as its occurrences in H. If $i=j=b+ks$, then occurrences in H_ω of $F_f(i, j)$ are cosigned with occurrences in H of $F_f(\alpha^m(a), \alpha^m(a))$,

of $F_f(\alpha^m(b), \alpha^m(b))$, of $F_f(\beta^p(a), \beta^p(a))$, or of $F_f(\beta^p(b), \beta^p(b))$ for all $m, p \geqq 0$. Hence by (Δ) above, all occurrences of $F_f(i, j)$ in H_ω are cosigned. Finally, if $i = j = b + ks + r$ $(1 \leqq r \leqq s - 1)$, then all occurrences in H_ω of $F_f(i, j)$ are cosigned with occurrences in H of $F_f(\alpha^m(a+r), \alpha^m(a+r))$ or of $F_f(\beta^p(a+r), \beta^p(a+r))$ for all $m, p \geqq 0$. But again these latter atomic occurrences are all cosigned in H.

(2) Let $j=i+1$ or $i=j+1$. If $i<b$, then all occurrences of $F_f(i, j)$ have the same sign in both H and H_ω. If $i=b+ks$, then occurrences of $F_f(i, i+1)$ in H_ω are cosigned with occurrences in H of $F_f(a, a+1)$ or of $F_f(b, b+1)$. Again consistency. Similarly for occurrences of $F_f(i+1, i)$. If $i=b+ks+r$ $(1 \leqq r \leqq s-1)$, then occurrences in H_ω of $F_f(i, i+1)$ are cosigned with occurrences in H of $F_f(a+r, a+r+1)$. Again consistency and again a similar argument for $F_f(i+1, i)$.

(3) Let $i-j>1$. Then the number atom $F_f(i, j)$ occurs in just two number disjuncts in H_ω, say $D_d(i-1, i, j)$ and $D_{d'}(i, i+1, j)$. [Indeed, this is the basic reason for the solvability of the class J.] If $i \leqq b$, then all occurrences of $F_f(i, j)$ are cosigned in both H and H_ω. If $i=b+ks+1$, then $i-1=b+ks=b+(k-1)s+s$. Then all occurrences of $F_f(i, j)$ in $D_d(i-1, i, j)$ are cosigned with the occurrences in H of $F_f(b+1, \alpha^{k-1-g+1}(j-gs))$ and all occurrences of $F_f(i,j)$ in $D_{d'}(i, i+1, j)$ are cosigned with the occurrences in H of $F_f(a+1, \alpha^{k-g+1}(j-gs))$, where g is the least number $\geqq 0$ such that $j-gs<b$. Hence by (Γ) above, all occurrences in H_ω of $F_f(i, j)$ are cosigned. Finally, if $i=b+ks+t$ where $2 \leqq t \leqq s$, then all occurrences of $F_f(i, j)$ in H_ω are cosigned with the occurrences in H of $F_f(a+t, \alpha^{k-g+1}(j-gs))$, where g is the least number $\geqq 0$ such that $j-gs<b$.

(4) Let $j-i>1$. Again the number atom $F_f(i, j)$ occurs in just two number disjuncts in H_ω, $D_d(i-1, i, j)$ and $D_{d'}(i, i+1, j)$. If $j \leqq b$, then all occurrences of $F_f(i, j)$ are cosigned in both H and H_ω. If $j=b+ks+q$ where $1 \leqq q \leqq s$, then we must look at i. If $i<b$, then all occurrences in H_ω of $F_f(i, j)$ are cosigned with the occurrences in H of $F_f(i, \beta^{k+1}(a+q))$. If $i=b+ws$ where $0 \leqq w \leqq k$, then all occurrences of $F_f(i, j)$ in $D_d(i-1, i, j)$ are cosigned with the occurrences in H of $F_f(b, \beta^{k-w+1}(a+q))$ and all occurrences in $D_{d'}(i, i+1, j)$ are cosigned with the occurrences in H of $F_f(a, \beta^{k-w}(a+q))$. [Remember that $b=a+s$.] Hence by (γ) above, all occurrences in H_ω of $F_f(i, j)$ are cosigned. Finally, if $i=b+ws+p$ where $1 \leqq p \leqq s-1$, then all occurrences in H_ω of $F_f(i, j)$ are cosigned with the occurrences in H of $F_f(a+p, \beta^{k-(w+1)+1}(a+q))$. [Note that here $g=w+1$.]

Thus the subclass J of G is solvable. For we have just seen that if S is any member of J and the number of letter disjuncts in S is δ, then S has a model if and only if the set $\{M(0, 1, 0), \cdots, M(\epsilon, \epsilon+1, \epsilon)\}$

is truth-functionally consistent where $\epsilon = 2^8$. Moreover, it is easy to show that the letter atomic forms xy and uy (or yx and yu) in conjunction with any other form can give axioms of infinity. For example, if we add the form xx, the following schema has no finite model but is satisfied when we replace $F_1\eta x$ by $\eta > x$, $F_2\eta x$ by $\eta + 1 > x$:

$$\neg F_1xx \wedge (F_1xy \supset F_1uy) \wedge (F_2xy \equiv F_1uy) \wedge F_2xx.$$

Other examples are:

$$F_1ux \wedge (F_1xy \supset F_1uy) \wedge \neg F_2ux \wedge (F_2uy \equiv F_1xy).$$

$$\neg F_1xx \wedge (F_1yx \supset F_1yu) \wedge F_2xx \wedge (F_2yx \equiv F_1yu).$$

$$F_1ux \wedge (F_1yu \supset F_1yx) \wedge \neg F_2ux \wedge (F_2yx \equiv F_1yu).$$

The classes L and L^* contain no axioms of infinity and are therefore solvable. This can be deduced from [1]. A much simpler direct argument can also be given. It will be omitted here. The decision problem of the classes Q and Q^* remains unsettled. We have the following partial results. The classes $\{xy, yu\}$, $\{yx, uy\}$, with no additional forms at all, are solvable and indeed contain no axioms of infinity. We can delete xu, uu from Q, ux, uu from Q^* and the decision problem remains the same. If Q and Q^* did contain axioms of infinity, they would have to be of a more complex sort than those occurring in J and J^*. The available data obtained so far lead us to believe that Q and Q^* are solvable and indeed contain no axioms of infinity. The open questions may be stated thus: Is $\{xx, xy, yu\}$ with no additional forms, or $\{xx, ux, yy, xy, yu\}$ solvable? Does either contain an axiom of infinity?

References

1. Burton Dreben, *Solvable Surañyi subclasses: An introduction to the Herbrand theory*, Annals of the Harvard Computation Laboratory (1962).

2. A. S. Kahr, Edward F. Moore, and Hao Wang, *Entscheidungsproblem reduced to the AEA case*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 365–377.

Harvard University