INVARIANT SUBSPACES OF CERTAIN LINEAR OPERATORS

BY KY FAN[1]

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1. Following the investigations of Pontrjagin [6] and Iohvidov [3] on linear operators in a Hilbert space with an indefinite inner product, M. G. Kreĭn [5] proved the following theorem.

THEOREM (PONTRJAGIN-IOHVIDOV-KREĬN). *Let E be the Hilbert space of infinite complex sequences $x=\{x_i\}$ with convergent $\sum_{i=1}^{\infty}|x_i|^2$, with norm $\|x\|=(\sum_{i=1}^{\infty}|x_i|^2)^{1/2}$. Let n be a positive integer and let*

$$J_n(x) = \sum_{i=1}^{n} |x_i|^2 - \sum_{i=n+1}^{\infty} |x_i|^2$$

for $x=\{x_i\}\in E$. If a linear transformation $\phi: E\to E$ is continuous in the norm topology, and if

(1) $J_n(x)\geqq 0$ *implies* $J_n(\phi(x))\geqq J_n(x)$,

then there exists an n-dimensional linear subspace F of E such that: (i) $\phi(F)\subset F$; (ii) $J_n(x)\geqq 0$ *for* $x\in F$; (iii) *every eigenvalue of the restriction of ϕ on F is of absolute value* $\geqq 1$.

This theorem is stronger than a result which Iohvidov [3] derived from the fundamental theorem of [6]. Iohvidov's theorem is so related to Pontrjagin's fundamental theorem that either one can be obtained from the other by a transform analogous to the Cayley transform (see [4]). Pontrjagin's proof of his theorem uses delicate and rather complicated arguments. Kreĭn's proof of the theorem stated above is much simpler and consists of an ingenious application of the fixed point principle.

In the present note we shall prove two results similar to the Pontrjagin-Iohvidov-Kreĭn theorem but of much more general nature, on existence of invariant subspaces of certain linear operators. It will be seen that the Pontrjagin-Iohvidov-Kreĭn theorem can be derived from our Theorem 2. All topological vector spaces considered here are implicitly assumed to be real or complex topological vector spaces satisfying the Hausdorff separation axiom.

2. We shall need the following lemma which was proved in [2].

LEMMA. *Let X be a nonempty compact convex set in a topological vector*

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space. Let A be a closed subset of $X \times X$ with the following two properties:

(2) *$(x, x) \in A$ for every $x \in X$.*

(3) *For each $x \in X$, the set $\{y \in X: (x, y) \notin A\}$ is convex (or empty). Then there exists a point $x_1 \in X$ such that $(x_1, y) \in A$ for all $y \in X$.*

THEOREM 1. *Let $E = E_1 \times E_2$ be the product of two locally convex topological vector spaces E_1, E_2, of which E_1 is of finite dimension n. Let S be a set in E with the two properties:*

(4) *For each $u \in E_1$, the set $S(u) = \{v \in E_2: (u, v) \in S\}$ is compact and convex.*

(5) *There exists an n-dimensional linear subspace L of E such that $L \subset S$ and $\pi_1(L) = E_1$, where π_1 denotes the projection from $E = E_1 \times E_2$ onto E_1.*

Let $\phi: E \to E$ be a continuous linear transformation satisfying the following condition:

(6) *For every n-dimensional linear subspace L of E such that $L \subset S$ and $\pi_1(L) = E_1$, there is an n-dimensional linear subspace M of E such that $M \subset S$, $\pi_1(M) = E_1$ and $\phi(L) \subset M$.*

Then there exists an n-dimensional linear subspace F of E such that $F \subset S$, $\pi_1(F) = E_1$ and $\phi(F) \subset F$.

PROOF. Let $\mathfrak{L}(E_1, E)$ be the vector space of all linear transformations from E_1 into E. Let $\mathfrak{J}$ be the topology of simple convergence for $\mathfrak{L}(E_1, E)$ (see [1, Chap. III, p. 18]). Because E_1 is finite dimensional, $\mathfrak{J}$ is also the topology of bounded convergence. The locally convex topological vector space obtained by topologizing $\mathfrak{L}(E_1, E)$ with $\mathfrak{J}$ will be denoted by $\mathfrak{L}_{\mathfrak{J}}(E_1, E)$.

Denote by π_2 the projection from $E = E_1 \times E_2$ onto E_2. Let $\mathfrak{K}$ be the set of all those $\xi \in \mathfrak{L}(E_1, E)$ such that $\pi_1 \circ \xi$ is the identity mapping on E_1 and $\xi(E_1) \subset S$. In other words, $\mathfrak{K}$ is the set of all $\xi \in \mathfrak{L}(E_1, E)$ such that $(\pi_1 \circ \xi)(u) = u$ and $(\pi_2 \circ \xi)(u) \in S(u)$ for every $u \in E_1$. Since $S(u)$ is convex, $\mathfrak{K}$ is a convex set in $\mathfrak{L}(E_1, E)$. For each $\xi \in \mathfrak{K}$, the image $L = \xi(E_1)$ is an n-dimensional linear subspace of E such that $L \subset S$ and $\pi_1(L) = E_1$. Conversely, if L is an n-dimensional linear subspace of E such that $L \subset S$ and $\pi_1(L) = E_1$, then each $u \in E_1$ determines a unique point $\xi(u) \in L$ such that $\pi_1(\xi(u)) = u$; the so-defined ξ is in $\mathfrak{K}$ and $\xi(E_1) = L$. Hence $\{\xi(E_1): \xi \in \mathfrak{K}\}$ is the set of all n-dimensional linear subspaces L of E such that $L \subset S$ and $\pi_1(L) = E_1$. In particular, (5) asserts that $\mathfrak{K}$ is nonempty.

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of E_1. Let V be a balanced (i.e., "équilibré" in [1]) convex neighborhood of 0 in E_2. Since each $S(e_i)$ is compact, there is an $\epsilon > 0$ such that $\epsilon \cdot S(e_i) \subset V$ for $1 \leqq i \leqq n$. If $u = \sum_{i=1}^{n} \lambda_i e_i$ and $\sum_{i=1}^{n} |\lambda_i| \leqq \epsilon$, then for every $\xi \in \mathfrak{K}$ we have

$(\pi_2 \circ \xi)(u) = \sum_{i=1}^n \lambda_i(\pi_2 \circ \xi)(e_i) \in \sum_{i=1}^n \lambda_i S(e_i) \subset V$. Thus $\{\pi_2 \circ \xi : \xi \in \mathcal{K}\}$ is an equicontinuous set of linear transformations from E_1 into E_2. Since $\pi_1 \circ \xi$ is the identity mapping on E_1 for every $\xi \in \mathcal{K}$, it follows that $\mathcal{K}$ is an equicontinuous set in $\mathcal{L}(E_1, E)$. For each $u \in E_1$, the set $\{\xi(u) : \xi \in \mathcal{K}\}$ is contained in $\{u\} \times S(u)$ and therefore is relatively compact in E. Consequently the equicontinuous set $\mathcal{K}$ is relatively compact in $\mathcal{L}_{\mathfrak{J}}(E_1, E)$. It is easy to verify that $\mathcal{K}$ is closed in $\mathcal{L}_{\mathfrak{J}}(E_1, E)$. Hence $\mathcal{K}$ is a nonempty compact convex set in $\mathcal{L}_{\mathfrak{J}}(E_1, E)$.

Consider an arbitrary $\xi \in \mathcal{K}$ and let $L = \xi(E_1)$. By (6), there is an n-dimensional linear subspace M of E such that $M \subset S$, $\pi_1(M) = E_1$ and $\phi(L) \subset M$. Let $\eta \in \mathcal{K}$ be such that $\eta(E_1) = M$. For each $u \in E_1$ we have $(\phi \circ \xi)(u) \in \phi(L) \subset \eta(E_1)$, so there is a $u_1 \in E_1$ such that $(\phi \circ \xi)(u) = \eta(u_1)$. Since $(\pi_1 \circ \eta)(u_1) = u_1$, we have

$$\begin{aligned}(\eta \circ \pi_1 \circ \phi \circ \xi)(u) &= (\eta \circ \pi_1 \circ \eta)(u_1) \\ &= \eta(u_1) = (\phi \circ \xi)(u).\end{aligned}$$

Hence $\eta \circ \pi_1 \circ \phi \circ \xi = \phi \circ \xi$. Thus for every $\xi \in \mathcal{K}$, there exists an $\eta \in \mathcal{K}$ with $\phi \circ \xi = \eta \circ \pi_1 \circ \phi \circ \xi$.

We claim that there exists a $\hat{\xi} \in \mathcal{K}$ such that

$$\phi \circ \hat{\xi} = \hat{\xi} \circ \pi_1 \circ \phi \circ \hat{\xi}. \tag{7}$$

Let $\{p_\nu\}_{\nu \in I}$ be the set of all continuous seminorms on $\mathcal{L}_{\mathfrak{J}}(E_1, E)$ (see [1, Chap. II, pp. 93–97]). For each $\nu \in I$, let $\mathcal{B}_\nu$ denote the set of all $\xi \in \mathcal{K}$ satisfying $p_\nu(\phi \circ \xi - \xi \circ \pi_1 \circ \phi \circ \xi) = 0$. Then the existence of a $\hat{\xi} \in \mathcal{K}$ satisfying (7) is equivalent to $\bigcap_{\nu \in I} \mathcal{B}_\nu \neq \emptyset$. The function $\xi \to \phi \circ \xi - \xi \circ \pi_1 \circ \phi \circ \xi$ from $\mathcal{L}_{\mathfrak{J}}(E_1, E)$ into itself is easily seen to be continuous, so each $\mathcal{B}_\nu$ is a closed subset of $\mathcal{K}$. By compactness of $\mathcal{K}$, in order to show $\bigcap_{\nu \in I} \mathcal{B}_\nu \neq \emptyset$, it suffices to prove that $\bigcap_{j=1}^k \mathcal{B}_{\nu_j} \neq \emptyset$ for every finite subset $\{\nu_1, \nu_2, \cdots, \nu_k\}$ of I. Given $\{\nu_1, \nu_2, \cdots, \nu_k\} \subset I$, let $\mathcal{A}$ denote the set of all $(\xi, \eta) \in \mathcal{K} \times \mathcal{K}$ satisfying

$$\sum_{j=1}^k p_{\nu_j}(\phi \circ \xi - \xi \circ \pi_1 \circ \phi \circ \xi) \leqq \sum_{j=1}^k p_{\nu_j}(\phi \circ \xi - \eta \circ \pi_1 \circ \phi \circ \xi).$$

One verifies easily that $(\xi, \eta) \to \phi \circ \xi - \eta \circ \pi_1 \circ \phi \circ \xi$ is a continuous function from $\mathcal{L}_{\mathfrak{J}}(E_1, E) \times \mathcal{L}_{\mathfrak{J}}(E_1, E)$ into $\mathcal{L}_{\mathfrak{J}}(E_1, E)$; so $\mathcal{A}$ is a closed subset of $\mathcal{K} \times \mathcal{K}$. Clearly $(\xi, \xi) \in \mathcal{A}$ for every $\xi \in \mathcal{K}$. Since the seminorms are convex functions, for every $\xi \in \mathcal{K}$ the set $\{\eta \in \mathcal{K} : (\xi, \eta) \notin \mathcal{A}\}$ is convex (or empty). By our Lemma there exists a $\xi_1 \in \mathcal{K}$ such that $(\xi_1, \eta) \in \mathcal{A}$ for all $\eta \in \mathcal{K}$. As we have seen above, there exists $\eta_1 \in \mathcal{K}$ satisfying $\phi \circ \xi_1 = \eta_1 \circ \pi_1 \circ \phi \circ \xi_1$. This equation and $(\xi_1, \eta_1) \in \mathcal{A}$ imply that $p_{\nu_j}(\phi \circ \xi_1 - \xi_1 \circ \pi_1 \circ \phi \circ \xi_1) = 0$ for $1 \leqq j \leqq k$, i.e., $\xi_1 \in \bigcap_{j=1}^{k} \mathcal{B}_{\nu_j} \neq \emptyset$. This proves the existence of a $\hat{\xi} \in \mathcal{K}$ satisfying (7).

Finally, let $F = \hat{\xi}(E_1)$. Then dim $F = n$, $F \subset S$ and $\pi_1(F) = E_1$. By (7) we have $\phi(F) = (\phi \circ \hat{\xi})(E_1) = (\hat{\xi} \circ \pi_1 \circ \phi \circ \hat{\xi})(E_1) \subset \hat{\xi}(E_1) = F$, which concludes the proof.

3. In the next theorem, we are interested in linear subspaces F which are not only invariant under ϕ but satisfy $\phi(F) = F$.

THEOREM 2. *Let $E = E_1 \times E_2$ be the product of two locally convex topological vector spaces E_1, E_2, of which E_1 is of finite dimension n. Let S be a set in E with the properties* (4), (5). *Let ϕ: $E \to E$ be a continuous linear transformation satisfying the following condition*:

(8) *For every n-dimensional linear subspace L of E such that $L \subset S$ and $\pi_1(L) = E_1$, we have* dim $\phi(L) = n$ *and* $\phi(L) \subset S$.

Then there exists an n-dimensional linear subspace F of E such that $F \subset S$, $\pi_1(F) = E_1$ and $\phi(F) = F$.

PROOF. Let L be an n-dimensional linear subspace of E such that $L \subset S$ and $\pi_1(L) = E_1$. Suppose $x \in L$ and $(\pi_1 \circ \phi)(x) = 0$. Then for every scalar α we have $\phi(\alpha x) \in \phi(L) \subset S$ and $(\pi_2 \circ \phi)(\alpha x) \in S((\pi_1 \circ \phi)(\alpha x)) = S(0)$. As $S(0)$ is compact and contains $\alpha \cdot (\pi_2 \circ \phi)(x)$ for every scalar α, we must have $(\pi_2 \circ \phi)(x) = 0$ and therefore $\phi(x) = 0$. This shows that $L \cap \operatorname{Ker}(\pi_1 \circ \phi) = L \cap \operatorname{Ker} \phi$. By (8), dim $\phi(L) = n$, so $L \cap \operatorname{Ker} \phi = \{0\}$. Hence $L \cap \operatorname{Ker}(\pi_1 \circ \phi) = \{0\}$, which means $\dim(\pi_1 \circ \phi)(L) = \dim L$, i.e., $(\pi_1 \circ \phi)(L) = E_1$. Thus, for every linear subspace L of E such that dim $L = n$, $L \subset S$ and $\pi_1(L) = E_1$, $\phi(L)$ again has these properties. Therefore condition (6) of Theorem 1 is satisfied. By Theorem 1, there exists an n-dimensional linear subspace F of E such that $F \subset S$, $\pi_1(F) = E_1$ and $\phi(F) \subset F$. But dim $\phi(F) = n$, so $\phi(F) = F$.

COROLLARY 1. *Let $E = E_1 \times E_2$ and S be the same as in Theorem 2. Let ϕ: $E \to E$ be a continuous linear transformation. If $\phi(S) \subset S$ and no one-dimensional linear subspace is contained in $S \cap \operatorname{Ker} \phi$, then there exists an n-dimensional linear subspace F of E such that $F \subset S$, $\pi_1(F) = E_1$ and $\phi(F) = F$.*

COROLLARY 2. *Let a Banach space $E = E_1 \times E_2$ be the product of two Banach spaces, of which E_1 is of finite dimension n and E_2 is reflexive. For $x = (u, v) \in E_1 \times E_2$, let*

$$q(x) = \|u\|_1 - \|v\|_2, \tag{9}$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in E_1 and E_2 respectively. If ϕ: $E \to E$ is a continuous linear transformation such that

(10) $x \neq 0$ *and* $q(x) \geqq 0$ *imply* $\phi(x) \neq 0$ *and* $q(\phi(x)) \geqq 0$,
then there exists an n-dimensional linear subspace F *of* E *such that* $\phi(F) = F$ *and* $q(x) \geqq 0$ *for all* $x \in F$. *If* ϕ *satisfies the stronger condition*:
(11) $x \neq 0$ *and* $q(x) \geqq 0$ *imply* $q(\phi(x)) > q(x)$,
then every eigenvalue of the restriction of ϕ *on* F *is of absolute value* >1.

PROOF. In the weak topology of E, ϕ remains to be continuous. Let $S = \{x \in E: q(x) \geqq 0\}$. Since E_2 is reflexive, for every $u \in E_1$, $S(u) = \{v \in E_2: \|v\|_2 \leqq \|u\|_1\}$ is convex and weakly compact. Condition (10) asserts that $\phi(S) \subset S$ and $S \cap \operatorname{Ker} \phi = \{0\}$. By Corollary 1 there exists an n-dimensional linear subspace F of E such that $\phi(F) = F$ and $q(x) \geqq 0$ for all $x \in F$. If, in addition, ϕ satisfies the stronger condition (11), and if $0 \neq x \in F$ and $\phi(x) = \lambda x$, then we have $|\lambda| q(x) = q(\lambda x) = q(\phi(x)) > q(x) \geqq 0$, whence $|\lambda| > 1$.

It is clear that Corollary 2 is valid for many functions q other than that defined by (9). For instance, we could have used $q(x) = \|u\|_1^2 - \|v\|_2^2$. With this q in the case of a Hilbert space E, Corollary 2 becomes the Pontrjagin-Iohvidov-Kreĭn theorem except that hypothesis (1) is replaced by the stronger condition:
(12) $x \neq 0$ and $J_n(x) \geqq 0$ imply $J_n(\phi(x)) > J_n(x)$.
In Kreĭn's proof, the theorem was first established under this stronger hypothesis (12), the general case was easily accomplished by approximating ϕ by linear transformations satisfying (12).

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NORTHWESTERN UNIVERSITY