ON DIFFERENTIABLE IMBEDDINGS OF SIMPLY-CONNECTED MANIFOLDS

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1. **Introduction.** We will be concerned with the problem of imbedding (differentiably) a closed *simply-connected* n-manifold M in the m-sphere S^m. According to [3], this problem depends only upon the homotopy type of M, in a "stable" range of dimensions. We obtain an explicit equivalent homotopy problem.

We also consider the problem of determining whether two imbeddings of M in S^m are isotopic (see [3] for basic definitions). A "homotopy condition" for deciding this question will also be obtained, again in a "stable" range of dimensions.

All manifolds, imbeddings and isotopies are to be differentiable. If M, V are manifolds with boundary and f is an imbedding of M in V, it will always be understood that $f(M)\cap\partial V=f(\partial M)$ and the intersection is transverse.

2. **Imbedding theorem.** M will, hereafter, denote a closed simply-connected n-manifold, $n>4$. Suppose f imbeds M in S^m; then we can define the normal plane bundle ν_f and, by a construction of Thom [10], an element $\alpha_f\in\pi_m(T(\nu_f))$, where $T(\nu_f)$ is the *Thom space* (see [10]) of ν_f. We call the pair (ν_f, α_f) the *normal invariants* of f. The existence of an imbedding, in particular, implies the existence of an $(m-n)$-plane bundle ξ whose Thom space is *reducible* in the sense of [1]. It follows from [1] that this property of M is a homotopy invariant and such a bundle ξ must be, a priori, stably fiber homotopy equivalent to the stable normal bundle of M.

Let M_0 denote the complement of an open disk in M.

THEOREM 1. *Suppose $2m\geqq 3(n+1)$ and ξ is an $(m-n)$-plane bundle over M stably equivalent to the stable normal bundle of M, such that $T(\xi)$ is reducible. Then there is an imbedding f of M in S^m such that ν_f is fiber homotopy equivalent to ξ:*

(a) *Over M if $n=6$, 14 or $n\not\equiv 2$* mod 4.

(b) *Over M_0 if $n\equiv 2$* mod 4.

It is to be expected that the conclusion of (a) is valid for all n. The difficulty in the proof arises from the lack of a satisfactory general definition of the *Arf invariant* (see [7]). In certain special cases, e.g., if M is a π-manifold or $\pi_i(M)=0$ for $2i<n$, we can obtain the conclusion of (a).

3. **Isotopy theorems.** Suppose f, g are isotopic imbeddings of M in S^m. It is easy to show that there is a bundle map $\phi\colon \nu_f \to \nu_g$ (note that the terminology implies that ϕ covers the identity map of M) such that $\phi_*(\alpha_f) = \alpha_g$. We say ϕ induces an *equivalence* between the normal invariants of f and g.

THEOREM 2. *Suppose $2m > 3(n+1)$. Then two imbeddings of M in S^m are isotopic if and only if they have equivalent normal invariants.*

Theorems 1 and 2 represent alternatives to the classification theorems of [5]. Note that the normal bundle plays a more prominent role here; in particular, Theorem 1 gives us information on the possible normal bundles of imbeddings.

The situation is more complicated in the borderline case $2m = 3(n+1)$. For $n = 4k-1$, $m = 6k$, we obtain a generalization of the main result of [4]. Let (ξ, α) be the normal invariants of an imbedding of M in S^m; we will define a cyclic group $Z(\xi, \alpha)$. If f, g are imbeddings of M in S^m whose normal invariants are equivalent to (ξ, α) we define a further invariant $L(f, g) \in Z(\xi, \alpha)$.

For the following theorem we must impose an additional restriction upon M:

(*) If H is a homotopy n-sphere which bounds a π-manifold and the connected sum $M \# H$ is diffeomorphic to M, then H is diffeomorphic to S^n.

THEOREM 3. *Suppose $n = 4k-1$, $m = 6k$.*

(a) *If f, g are imbeddings of M in S^m with equivalent normal invariants, then f and g are isotopic if and only if $L(f, g) = 0$.*

(b) *If f is an imbedding with normal invariants (ξ, α) and $L \in Z(\xi, \alpha)$, then there exists an imbedding g whose normal invariants are equivalent to (ξ, α) such that $L(f, g) = L$.*

Thus $L(f, g)$ plays the role of a difference cochain in obstruction theory. One may conjecture on the existence of a higher obstruction theory for imbeddings with equivalent normal invariants in the "nonstable" range of dimensions.

4. **Discussion of proofs.** We use a nonstable version of the procedures introduced in [2; 9] (see [8] for details). To prove Theorem 1 we construct a submanifold N of S^m, and a map $h\colon N \to M$ of degree $+1$ such that $h^*\xi$ is the normal bundle of N in S^m. By a suitable generalization of the techniques in [4, §3], we can perform spherical modifications on the pair (S^m, N), at each stage defining a new map h so that $h^*\xi$ is still the normal bundle. We must use the restriction

on codimension here. Following [2; 8] we can eventually make h a homotopy equivalence, if $n=6$, 14 or $n \not\equiv 2 \bmod 4$. If $n \equiv 2 \bmod 4$, we must replace S^m by the m-disk D^m and let N be a bounded manifold imbedded in D^m, with ∂N a homotopy sphere and $h: N \to M_0$ such that $h^*(\xi | M_0)$ is the normal bundle to N in D^m. Now we can perform spherical modifications as above to make h a homotopy equivalence. Using the results of [3; 6] one can deform a homotopy inverse of h into an imbedding of M (or M_0) into S^m (or D^m) with a normal bundle fibre homotopy equivalent to ξ (or $\xi | M_0$). If $n \equiv 2 \bmod 4$, the imbedding of M_0 into D^m can be extended to M into S^m.

Theorem 2 is approached by a similar combination of the methods of [9] (see [8]) and the techniques of [4]. At one point we need the following result, which follows easily, in this range of dimensions, from the results of [3]. Let H be a homotopy n-sphere and $g: M \to M \# H$ a diffeomorphism homotopic to the standard homeomorphism. Let f be an imbedding of M in S^m, i an imbedding of H in S^m and f' the imbedding (unique up to isotopy) of $M \# H$ in S^m induced by f and i. Then $f' \cdot g$ is isotopic to f.

The proof of Theorem 3 proceeds as that of Theorem 2 up to a point. In order to complete the necessary spherical modifications we must consider a linking number invariant (an integer) similar to that defined in [4]. After reducing to a quotient group, $Z(\xi, \alpha)$, we obtain $L(f, g)$ which depends only on f and g. Now, the verification of (a) is not unlike the arguments in [4, §3]; (b) is proved by adjoining to f one of the knotted spheres constructed in [4].

5. **More general results.** Given a closed m-manifold V and $v \in H_n(V)$, we may ask whether v can be realized by an imbedding f of M. If so, we can define the normal bundle ν_f and $\alpha_f \in \pi(V, T(\nu_f))$ (= homotopy classes of maps $V \to T(\nu_f)$), using the procedures of [10], such that $\alpha_f^*(u(\nu_f)) =$ dual of v (if ξ is a k-plane bundle over M, $u(\xi) \in H^k(T(\xi))$ is the usual generator). Also $f^*\tau_V = \tau_M + \nu_f$, where τ_M denotes the tangent bundle of M. In particular, if $n=4k$ and $p_i(V)=0$ for $0<i<k$:

$$(\dagger) \qquad \text{index } M = L_k(\bar{p}_1(\xi), \cdots, \bar{p}_{k-1}(\xi), \bar{p}_k(\xi) + \langle p_k(V), v \rangle M)$$

where $\bar{p}_i(\xi)$ is the *dual* Pontryagin class of ξ, L_k is the Hirzebruch polynomial (see [7]) and $\xi = \nu_f$.

THEOREM 4. *Let M, V, v be as above and $2m \geqq 3(n+1)$. Assume $\pi_i(V)=0$ for $2i \leqq n$. Suppose ξ is an $(m-n)$-plane bundle over M satisfying (†) if $n=4k$ and there exists $\alpha \in \pi(V, T(\xi))$ such that $\alpha^*(u(\xi))$*

= dual of v. Then there is an imbedding f of M in V, representing v, such that ν_f *is fiber homotopy equivalent to* ξ:

(a) *Over* M *if* $n = 6$, 14 *or* $n \not\equiv 2 \bmod 4$.

(b) *Over* M_0 *if* $n \equiv 2 \bmod 4$.

The proof is similar to that of Theorem 1. There is also an isotopy theorem.

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