RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON FLOQUET'S THEOREM IN HILBERT SPACES

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Communicated by H. A. Antosiewicz, October 22, 1963

We are concerned with the question of extending the validity of Floquet's Theorem on linear periodic differential equations to equations in a Hilbert space. This question was raised and discussed (for equations in a Banach space) in [2]. Further results for Banach spaces will appear elsewhere.

Specifically, we consider, in a real or complex Hilbert space X, the equation

$$U + AU = 0, \tag{1}$$

where A is an operator-valued, locally (Bochner) integrable function of the real variable t, periodic with period 1 (for the sake of normalization); we denote by U the unique operator-valued solution of (1) that satisfies $U(0)=I$, where I is the identity operator. We set $U_0 = U(1)$.

We say that there is a *Floquet representation of order m* (m a positive integer) if there exists an operator B such that the operator-valued function P defined by $P(t) = U(t)e^{tB}$ is periodic with period m. It is easy to see that this is equivalent to the existence of a logarithm of U_0^m [2, Lemma 2.1].

The classical Floquet Theorem states that, if X is finite-dimensional, there is a Floquet representation of order 1, if the space is complex, and of order at most 2, if the space is real (see, e.g., [1, pp. 78, 81, 106–107]). It was shown in [2, Example 2.1] that if X is infinite-dimensional there need not be any Floquet representation at all; in that example, the space is real or complex, and separable, and A is continuous and differs by as little as we please (uniformly) from a constant of norm $=\pi$, so that $\int_0^1 \|A(t)\| dt$ exceeds π by as little as we please. On the other hand, it was shown in [2, Theorem 2.1] (by Banach space methods) that if $\int_0^1 \|A(t)\| dt < \log 4$, there always exists a Floquet representation of order 1. We shall now almost fill this gap between log 4 and π:

THEOREM. *If* $\int_0^1\|A(t)\|dt<\pi$, *there exists a Floquet representation of order* 1.

Using only Banach space methods, we connect the existence of the representation with properties of U_0:

LEMMA. *If* $U_0+\sigma I$ *is invertible for every real* $\sigma\geqq 0$, *there exists a Floquet representation of order* 1.

PROOF. Under the assumption, I and U_0 are connected by the arc of invertible elements $(1-\lambda)I+\lambda U_0$, $0\leqq\lambda\leqq 1$, in the (commutative) closed subalgebra of operators generated by U_0. By a theorem of Nagumo (see [3, Theorem (1.4.12)]), U_0 has a logarithm in the same subalgebra; the conclusion follows.

The proper Hilbert space part of the proof of the Theorem requires some geometric preliminaries. For every $x\in X\setminus\{0\}$ we write $x=\|x\|\operatorname{sgn} x$. The angle between $x, y\in X\setminus\{0\}$ is denoted by $\theta(x, y)$, where $0\leqq\theta(x, y)\leqq\pi$, $\cos\theta(x, y)=\operatorname{Re}(\operatorname{sgn} x, \operatorname{sgn} y)$, the "Re" being redundant in the real case. If f, $\dot{f}$ are functions of t with values in X, such that $\dot{f}$ is locally (Bochner) integrable, and $f(t)=f(0)+\int_0^t\dot{f}(u)du\neq 0$ for all t, we say that f is a *nonvanishing primitive*. If f is a nonvanishing primitive, so are $\|f\|$ (with real values) and $\operatorname{sgn} f$, and indeed $\dot{f}=\|f\|^{\cdot}\operatorname{sgn} f+\|f\|(\operatorname{sgn} f)^{\cdot}$; the two terms being orthogonal for every t, we have

$$\|\dot{f}\|^2=(\|f\|^{\cdot})^2+\|f\|^2\|(\operatorname{sgn} f)^{\cdot}\|^2 \tag{2}$$

(here, as already in (1), equality or inequality of locally integrable functions is understood to hold a.e.).

PROOF OF THE THEOREM. We show that $U_0+\sigma I$ is invertible for all $\sigma\geqq 0$. The set of those σ for which this is true is open and contains 0; if it does not contain all $\sigma\geqq 0$, there is a boundary value $\sigma_0>0$; it is well known that there exists a sequence (x_n) in X such that $\|x_n\|=1$ and $y_n=(\sigma_0^{-1}U_0+I)x_n\to 0$.

Applying (2) to the nonvanishing primitive Ux_n and using (1),

$$\begin{aligned}\pi>\int_0^1\|A(t)\|dt &\geqq\int_0^1\frac{\|\dot{U}(t)x_n\|}{\|U(t)x_n\|}dt\\ &\geqq\int_0^1\|(\operatorname{sgn} U(t)x_n)^{\cdot}\|dt.\end{aligned} \tag{3}$$

Now the last member of (3) is the length of an arc on the surface of the unit sphere, connecting $\operatorname{sgn} x_n=x_n$ and $\operatorname{sgn} U_0x_n=\operatorname{sgn}(-x_n+y_n)$. Therefore

$$
\begin{aligned}
\int_0^1 \|(\operatorname{sgn} U(t)x_n)^{\cdot}\|dt &\geqq \theta(x_n, -x_n + y_n) \\
&= \pi - \theta(x_n, x_n - y_n) \qquad (4)\\
&\geqq \pi - \arcsin \|y_n\| \to \pi.
\end{aligned}
$$

Combining (3) and (4), we obtain a contradiction. The conclusion follows from the lemma.

The gap is now filled except for the case $\int_0^1\|A(t)\|dt=\pi$. (A more effective use of (2) shows that in that case $U_0+\sigma I$ is still invertible for all $\sigma\geqq 0$, except possibly for $\sigma=1$.) It is an open question whether in that case a Floquet representation of *some* order always exists if the dimension is infinite. For real X, however, there need not exist any of order 1, even when the dimension is 2, as the following example shows; for real X the inequality in the statement of the theorem is thus best possible.

EXAMPLE. In the real plane X we represent operators by matrices through some choice of cartesian coordinates. Consider

$$
A(t) = \frac{1}{2}\pi\begin{pmatrix} \sin 2\pi t & 1-\cos 2\pi t \\ -1-\cos 2\pi t & -\sin 2\pi t\end{pmatrix},
$$

a continuous function with period 1; a direct computation yields $\|A(t)\|=\pi$, a constant. It is easily verified that

$$
U(t) = \begin{pmatrix} \cos \pi t & \pi t\cos \pi t - \sin \pi t \\ \sin \pi t & \pi t \sin \pi t + \cos \pi t\end{pmatrix},
$$

so that

$$
U_0 = \begin{pmatrix} -1 & -\pi \\ 0 & -1 \end{pmatrix};
$$

this operator has no (real) square root, let alone a (real) logarithm.

REFERENCES

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