A COMPLETE CLASSIFICATION OF THE Δ_2^1-FUNCTIONS

BY YOSHINDO SUZUKI

Communicated by Leon Henkin, September 28, 1963

Suslin has shown that a set is a Borel set if and only if both it and its complement are analytic sets [14]. Kleene has proved an analogous theorem for the hyperarithmetical sets [7; 9]. Those hierarchies are so naturally constructed that we can establish significant propositions with the aid of them. A lot of effort was made to construct a natural hierarchy for B_2-sets. They are, however, incomplete and contain only a small portion of B_2-sets [10]. The situation was the same for the Δ_2^1-functions of the natural numbers[1] and, if we consider the reason why our trials failed [13; 18], we should say that some new principles were required to settle our problem. Shoenfield [20] constructed for the first time a complete hierarchical classification of the Δ_2^1-functions. Namely, he showed, by the aid of the effective version of the uniformization principle of Kondô [11; 1], that every Δ_2^1-function is constructible from a Δ_2^1-ordinal and conversely. Ours has the same character as his in the use of the uniformization principle. We shall define another classification and shall prove it to be complete by using that principle.[2] We shall study our classification in relation to the hyperdegree of Kleene and shall prove that it is neither fine nor coarse. Although we have not done so, comparison of the two complete classifications may be worthy of study.

CLASSIFICATION. Let γ be the unique solution of the condition $(\alpha)(Ex)P(\beta, \bar{\alpha}(x))$.[3] We shall then say that γ is defined by the sieve P and P is a sieve for γ. Let us denote by $\mathcal{U}$ the set of functions γ defined by recursive sieves. Let $T_{P,\beta}$ be the set of sequence numbers in $\overline{P}^{\langle\beta\rangle}$ which are neither secured nor past secured [8]. For any recursive sieve R, there is a recursive sieve Q for which the identity $T_{R,\beta} = T_{Q,\beta} = \overline{Q}^{\langle\beta\rangle}$ holds for every β. For γ in $\mathcal{U}$ we shall denote by $\tau(\gamma)$ the smallest of the ordinals $\tau(T_{R,\gamma})$ where R are recursive sieves for γ. $\mathcal{V}$ is the set of $\tau(\gamma)$ for γ in $\mathcal{U}$. If γ is in $\mathcal{U}$, γ is evidently a

[1] A problem of Tugué [24, p. 117] was negatively solved by him and us. It was also solved by Shoenfield [21] and Gandy [5].

[2] Theorem 1 is a precise formulation of a statement of Kondô's. See our Remark to Theorem 1.

[3] Notations are those of [6; 7; 8; 9]. Some notations are also borrowed from [11]. We shall use Σ_n^t, Π_n^t notation of [1]. Δ_n^t is the intersection of the Σ_n^t and Π_n^t families [20]. Following notations are used: $\langle x_0, \cdots, x_n\rangle$ for $p_0^{x_0} * \cdots * p_n^{x_n}$, $P^{\langle\mathfrak{a}\rangle}$ for the set of sequence numbers u for which $P(\mathfrak{a}, u)$.

Δ_2^1-function. Conversely to this we have the Theorem 1. In the proof of the Theorem 1 we shall make use of the

PRINCIPLE OF UNIFORMIZATION. Every Π_1^1-set can be made uniform by a Π_1^1-set [**11**; **1**; **19**].

THEOREM 1. *Every Δ_2^1-function γ is hyperarithmetical in some β_0 in the set* $\mathfrak{U}$.[4]

PROOF. By our hypothesis, there are recursive predicates S_i $(i=0, 1)$ for which

$$\begin{aligned}\gamma(y) = z &\equiv (E\beta)(\alpha)(Ex)S_0(y, z, \beta, \alpha, x)\\ &\equiv (\beta)(E\alpha)(x)\overline{S}_1(y, z, \beta, \alpha, x).\end{aligned}$$

By the uniformization principle, we may assume

$$(E\beta)(\alpha)(Ex)S_i(y, z, \beta, \alpha, x) \equiv (E!\beta)(\alpha)(Ex)S_i(y, z, \beta, \alpha, x).$$

As S_i are made uniform, for any $\langle y, z\rangle$, there is a uniquely determined function $\beta_{\langle y,z\rangle}$ for which

$$(\alpha)(Ex)S_i(y, z, \beta, \alpha, x) \rightarrow \beta = \beta_{\langle y,z\rangle}.$$

Let β_0 be defined by the condition:

$$\begin{aligned}\beta_0(\langle y, z, t\rangle) &= \beta_{\langle y,z\rangle}(t),\\ \beta_0(t) &= 1 \qquad \text{for } t \neq \langle (t)_0, (t)_1, (t)_2\rangle.\end{aligned}$$

There are recursive predicates R_i (7) 1.3 for which

$$(\alpha)(Ex)S_i(z, y, \lambda t\beta(\langle y, z, t\rangle), \alpha, x) \equiv (\alpha)(Ex)R_i(z, y, \beta, \alpha, x).$$

For those R_i,

$$(E\beta)(\alpha)(Ex)S_i(y, z, \beta, \alpha, x) \equiv (\alpha)(Ex)R_i(y, z, \beta_0, \alpha, x).$$

Consequently γ is hyperarithmetical in β_0. On the other hand, β_0 is in $\mathfrak{U}$ as it is defined by the following condition:

$$\begin{aligned}&(t)[\langle (t)_0, (t)_1, (t)_2\rangle \neq t \rightarrow \beta(t) = 1] \;\&\\ &\qquad (y)(z)[(\alpha)(Ex)R_0(y, z, \beta, \alpha, x) \vee (\alpha)(Ex)R_1(y, z, \beta, \alpha, x)].\end{aligned}$$

REMARK. Our Theorem 1 is closely related to a result of Shoenfield [**20**, Theorem, p. 136]. By his Theorem all Δ_2^1-functions are well ordered naturally. If we apply a result of Addison [**2**] to his, we can see that every Δ_2^1-function is hyperarithmetical in a Δ_2^1-ordinal and conversely. We can see from this that the relation between the two is deep, and so comparison of the two classifications may throw light

[4] See [**9**, Theorem XXIV], [**7**, Theorem 5 and Theorem 9].

on the family of the Δ_2^1-functions. In this respect, the following question raised by the referee may be fundamental: Is our classification essentially different from that given by the natural ordering of the constructible sets, i.e., does $Od^{\prime}\beta \leqq Od^{\prime}\gamma$ always follow, for β, γ in $\mathfrak{U}$, from the proposition that the hyperdegree of β is lower than that of γ?

For $\nu \in \mathfrak{V}$, we denote by $\boldsymbol{C}_\nu$ the set of functions γ hyperarithmetical in some β_0 in $\mathfrak{U}$ for which $\tau(\beta_0)=\nu$. By our Theorem 1, $\bigcup_{\nu\in\mathfrak{V}} \boldsymbol{C}_\nu$ is identical to the family of the Δ_2^1-functions. Conversely the order ν of the class $\boldsymbol{C}_\nu$ corresponds to the complexity of their members. That is,

THEOREM 2. *For β_0, β_1 in $\mathfrak{U}$, $\tau(\beta_0) \leqq \tau(\beta_1)$ only if β_0 is hyperarithmetical in β_1.*

PROOF. Let R_i be recursive sieves for which $\tau(\beta_i)=\tau(\overline{R}_i^{\langle\beta_i\rangle})$. As $\tau(\beta_0) \leqq \tau(\beta_1)$ and β_0 is defined by the sieve R_0,

$$\begin{aligned}\beta = \beta_0 &\equiv (E\Psi)[\Psi \text{ is an isomorphism of } \overline{R}_0^{\langle\beta\rangle} \text{ into } \overline{R}_1^{\langle\beta_1\rangle}] \\ &\equiv (E\alpha)(x)\overline{S}(\beta, \beta_1, \alpha, x)\end{aligned}$$

where $S(\beta, \gamma, \alpha, x)$ is a recursive predicate. We see thus β_0 is hyperarithmetical in β_1 [9].

We shall give a theorem related to our Theorem 2.

THEOREM 3. *For β_0, β_1 in $\mathfrak{U}$, β_1 is hyperarithmetical in β_0 if and only if $\tau(\beta_1) < \omega_1^{\beta_0}$.*

PROOF. Let R_1 be a recursive sieve for β_1 for which $\tau(\beta_1)=\tau(\overline{R}_1^{\langle\beta_1\rangle})$. If $\tau(\beta_1) < \omega_1^{\beta_0}$, then there is a partial recursive predicate $m <^{\beta_0} n$ recursive in β_0 whose order type is that of $\tau(\beta_1)$ [8, Proposition A]. For this predicate,

$$\begin{aligned}\beta = \beta_1 &\equiv (E\Psi)[\Psi \text{ is an isomorphism of } \overline{R}_1^{\langle\beta_0\rangle} \text{ into } \lambda mn\, m <^{\beta_0} n] \\ &\equiv (E\alpha)(x)\overline{S}(\beta, \beta_0, \alpha, x) \\ &\qquad (S(\beta, \gamma, \alpha, x) \text{ is partial recursive and} \\ &\qquad\quad \lambda\beta\alpha x\, S(\beta, \beta_0, \alpha, x) \text{ is completely defined}) \\ &\equiv (E\alpha)(x)\overline{R}(\beta, \beta_0, \alpha, x) \\ &\qquad (R(\beta, \gamma, \alpha, x) \text{ is recursive by [7, Lemma 1]}).\end{aligned}$$

As in the proof of Theorem 2, β_1 is hyperarithmetical in β_0. Conversely, let β_1 be hyperarithmetical in β_0. Evidently $\beta = \beta_1$ is a Σ_1^1-predicate in β_0. The well ordered relation $<\cdot$ on $\overline{R}_1^{\langle\beta_1\rangle}$ is reduced in the following way:

$$\begin{aligned} s <\cdot\, r &\equiv [s \prec r \;\&\; s, r \in \overline{R}_1^{\langle \beta_1 \rangle}] \\ &\equiv (E\beta)[\beta = \beta_1 \;\&\; s \prec r \;\&\; s, r \in \overline{R}_1^{\langle \beta \rangle}] \\ &\equiv (E\alpha)(x)\overline{S}(s, r, \beta_0, \alpha, x) \\ &\qquad (S \text{ recursive, as } \beta = \beta_1 \text{ is } \Sigma_1^1 \text{ in } \beta_0). \end{aligned}$$

The relation $<\cdot$ is a Σ_1^1-well ordering in β_0 and consequently its order type $\tau(\beta_1)$ is $<\omega_1^{\beta_0}$ [12, p. 246].[5]

By our Theorem 2, every two elements of $\mathfrak{U}$ are comparable with respect to hyperdegree. If we use this fact with the Theorem 3, we have a corollary which shows our classification is not too fine.

COROLLARY 1. *The following three conditions are equivalent for* β_0, β_1 *in* $\mathfrak{U}$:

$$\beta_0 < \beta_1, \qquad \omega_1^{\beta_0} < \omega_1^{\beta_1}, \qquad F[\beta_0] \leqq \beta_1.^6$$

PROOF. If $\beta_0<\beta_1$, then $\omega_1^{\beta_0}\leqq\tau(\beta_1)$ by our Theorem 3 and then $\omega_1^{\beta_0}<\omega_1^{\beta_1}$ by [15, Theorem 6]. Let us assume $\omega_1^{\beta_0}<\omega_1^{\beta_1}$. By [22, Corollary 6.2] and by our Theorem 2, $\beta_0<\beta_1$ and by [22, Corollary 6.1] we have $F[\beta_0]\leqq\beta_1$. The last implication is immediate from [7, Theorem 4] and the transitivity of the hyperdegree [9, p. 210].

By our Corollary 1 and [4, Theorem 1] we have the

COROLLARY 2. *For some* Δ_2^1-*function* γ, γ *is not in the set* $\mathfrak{U}$.[7]

THE SET $\mathfrak{U}$. We have defined a complete classification $\bigcup_{\nu\in\mathfrak{V}} C_\nu$ of the Δ_2^1-functions and have showed how those subclasses C_ν are related to each other. We shall give in this section some examples of Δ_2^1-functions which belong to $\mathfrak{U}$. From those examples, we may be allowed to say our classification is not a coarse one. We have to use in the proof of Theorem 4 the

ISOMORPHISM THEOREM. *There is a partial recursive functional* $M[\phi, \psi]$ *with the following properties* [16, *Theorem* 18]:

1°. *If* ϕ *and* ψ *are* 1-1 *functions, then* $\lambda x M[\phi, \psi](x)$ *is completely defined.*

2°. *If* α *and* β *are* 1-1 *equivalent*[8] *with respect to* ϕ *and* ψ, *then* α *and* β *are isomorphic with respect to* $M[\phi, \psi]$.

[5] Mr. H. Tanaka called our attention to the fact that the method of [12] can be used to show that every Σ_1^1-well ordering represents a constructive ordinal.

[6] $\leqq$ and $<$ are used for hyperdegree. $F[\beta]$ is the representing function of the predicate $\lambda a(\alpha)(Ex)T_1^{\beta,1}(\bar{\alpha}(x), a, a)$.

[7] We can use the principle of uniformization [23, Theorem 2], and our Theorem 2 instead of [4, Theorem 1] and our Corollary 1.

[8] Although those notions were defined for the sets of natural numbers [17; 16] they can be extended to the functions of natural numbers.

Conversely to the Corollary 1, we have the

THEOREM 4. *If* β_0 *is in* $\mathfrak{U}$, *then* $F[\beta_0]$ *is also in* $\mathfrak{U}$.

PROOF. Let $\gamma^\beta(t)$ be the representing predicate of the relation $\lambda t[(t)_0 <_0^\beta (t)_1 \ \& \ t = \langle (t)_0, (t)_1\rangle]$. There is a recursive predicate $Q(\delta, \beta, w, v, u)$ [25] such that γ^β is uniquely defined by the condition:

$$(1) \quad \begin{aligned} &(t)\gamma(t) \leqq 1 \ \& \ (w)(Ev)(u)\overline{Q}(\gamma, \beta, w, v, u) \ \& \\ &(\delta)[(w)(Ev)(u)\overline{Q}(\delta, \beta, w, v, u) \rightarrow (t)(\gamma(t) = 0 \rightarrow \delta(t) = 0)]. \end{aligned}$$

As was proved in [7, Lemma 6], there is a recursive function $\nu_1(a)$ for which

$$(2) \quad \begin{aligned} \beta((a)_0) = (a)_1 &\equiv \nu_1(a) \in O^\beta \\ &\equiv \gamma^\beta(\langle 1, 2 \exp \nu_1(a)\rangle) = 0. \end{aligned}$$

Let $\rho(\gamma, a)$ be the partial recursive function $\mu t\, \gamma(\langle 1, 2 \exp \nu_1(\langle a, t\rangle)\rangle)$. Evidently $\beta = \lambda a\, \rho(\gamma^\beta, a)$. As β_0 is in $\mathfrak{U}$, there is a recursive sieve R for β_0. By [6, Theorems II, VI] and [7, Lemma 1], there is a recursive predicate $R_1(\gamma, \alpha, x)$ such that

$$(3) \quad (Ex)R(\lambda a\, \rho(\gamma, a), \alpha, x) \equiv (Ex)R_1(\gamma, \alpha, x)$$

for γ and α for which $\lambda x R(\lambda a \rho(\gamma, a), \alpha, x)$ is completely defined. As in the case for R, there is a recursive predicate $Q_1(\delta, \gamma, w, v, u)$ such that

$$(4) \quad (u)\overline{Q}(\delta, \lambda a\, \rho(\gamma, a), w, v, u) \equiv (u)\overline{Q}_1(\delta, \gamma, w, v, u)$$

for δ and γ for which $\lambda u \overline{Q}(\delta, \lambda a\, \rho(\gamma, a), w, v, u)$ is completely defined. If $\lambda a\, \rho(\gamma, a)$ is completely defined, those requirements are clearly met. We shall show that the function γ^{β_0} is uniquely defined by the condition $(\alpha)(Ex)S_1(\gamma, \alpha, x)$ with S_1 recursive:

$$(5) \quad \begin{aligned} &(t)\gamma(t) \leqq 1 \ \& \ [\lambda a\, \rho(\gamma, a) \text{ is completely defined}] \ \& \\ &(\alpha)(Ex)R_1(\gamma, \alpha, x) \ \& \ (w)(Ev)(u)\overline{Q}_1(\gamma, \gamma, w, v, u) \ \& \\ &(\delta)[(w)(Ev)(u)\overline{Q}_1(\delta, \gamma, w, v, u) \rightarrow (t)(\gamma(t) = 0 \rightarrow \delta(t) = 0)]. \end{aligned}$$

Let us assume $\gamma = \gamma^{\beta_0}$. By the equivalences (2), (3) and (4),

$$\begin{aligned} &[\lambda a\, \rho(\gamma, a) \text{ is completely defined}] \ \& \ \lambda a\, \rho(\gamma, a) = \beta_0 \ \& \\ &[(\alpha)(Ex)R_1(\gamma, \alpha, x) \equiv (\alpha)(Ex)R(\beta_0, \alpha, x)] \ \& \\ &(\delta)[(w)(Ev)(u)\overline{Q}_1(\delta, \gamma, w, v, u) \equiv (w)(Ev)(u)\overline{Q}(\delta, \beta_0, w, v, u)]. \end{aligned}$$

As β_0 is defined by the sieve R and γ^{β_0} is the solution of the condition

(1), we see γ satisfies the condition (5). Conversely let γ be a solution of the condition (5). As $\lambda a\ \rho(\gamma, a)$ is completely defined,

$$(\alpha)(Ex)R_1(\gamma, \alpha, x) \equiv (\alpha)(Ex)R(\lambda a\ \rho(\gamma, a), \alpha, x)$$

and consequently $\beta_0 = \lambda a\ \rho(\gamma, a)$. We can now see γ is identical to γ^{β_0}. We have proved thus γ^{β_0} is in $\mathfrak{U}$.

Let us now show that the function $F[\beta_0]$ is in $\mathfrak{U}$. By [7, Theorem 4] and [8, Theorem II] both with uniformity in β, there are 1-1 recursive functions $\phi(a)$ and $\phi_1(a)$ for which

$$(6) \qquad \begin{aligned} \gamma^{\beta}(a) &= F[\beta](\phi(a)), \\ \beta((a)_0) = (a)_1 &\equiv F[\beta](\phi_1(a)) = 0. \end{aligned}$$

By [8, Theorem I] with uniformity in β and footnote 28, there is a recursive function $\xi_1(\beta, a)$ which is 1-1 for every β and for which

$$F[\beta](a) = \gamma^{\beta}(\langle 1, 2 \exp \xi_1(\beta, a)\rangle).$$

Let $\eta(\beta, a)$ be the partial recursive function

$$M[\lambda t\phi(t), \lambda t\langle 1, 2 \exp \xi_1(\beta, t)\rangle](a)$$

and $\rho_2(\delta, a)$ be the partial recursive function $\mu t\ \delta(\phi_1(\langle a, t\rangle)) = 0$. By the isomorphism theorem, $\eta(\beta, a)$ is general recursive and

$$(7) \qquad \gamma^{\beta}(a) = F[\beta](\eta(\beta, a)).$$

In the same way as R_1 and Q_1 were constructed from R and Q, we can construct recursive predicates R_2 and S_2 such that

$$(8) \qquad \begin{aligned} &(\alpha)(Ex)R(\lambda a\ \rho_2(\delta, a), \alpha, x) \equiv (\alpha)(Ex)R_2(\delta, \alpha, x), \\ &(\alpha)(Ex)S_1(\lambda a\ \delta(\eta(\lambda t\ \rho_2(\delta, t), a)), \alpha, x) \equiv (\alpha)(Ex)S_2(\delta, \alpha, x) \end{aligned}$$

for every δ for which $\lambda t\ \rho_2(\delta, t)$ is completely defined. We shall see $F[\beta_0]$ is defined by the following condition $(\alpha)(Ex)S(\delta, \alpha, x)$ with S recursive:

$$(9) \qquad \begin{aligned} &[\lambda a\ \rho_2(\delta, a) \text{ is completely defined}]\ \& \\ &(\alpha)(Ex)R_2(\delta, \alpha, x)\ \&\ (\alpha)(Ex)S_2(\delta, \alpha, x). \end{aligned}$$

Let us assume that δ be a solution of the condition (9). By the equivalence (6), we see $\lambda a\ \rho_2(\delta, a) = \beta_0$ and then

$$(\alpha)(Ex)S_1(\lambda a\ \delta(\eta(\beta_0, a)), \alpha, x)$$

by (8). As γ^{β_0} is defined by the sieve S_1 and $\eta(\beta_0, a)$ is a permutation of natural numbers, $\lambda a\ \delta(\eta(\beta_0, a)) = \gamma^{\beta_0}$ and consequently $\delta = F[\beta_0]$. The converse implication can be proved by using (6), (7) and (8).

Let $<^{\gamma}=[\lambda yz\,\gamma(\langle y, z\rangle)=0]$ be a well ordering. The $\mathfrak{H}$-completion π of γ [3] is defined by the following condition:

$$(t)\pi(t) \leqq 1 \;\&\; (t)[\pi(t) = 0 \rightarrow t = \langle (t)_0, (t)_1\rangle \;\&\; (t)_0 \leqq 1] \;\&$$
$$(t)\pi(\langle 0, t\rangle) = \gamma(t) \;\&\; (t)[\pi(\langle 1, t\rangle) = 0 \rightarrow t = \langle (t)_0, (t)_1\rangle \;\&$$
$$((t)_0 \text{ is in the field of } <^{\gamma})] \;\&$$
$$(u)(t)[(u \text{ is the first element of } <^{\gamma}) \rightarrow \pi(\langle 1, \langle u, t\rangle\rangle) = 0] \;\&$$
$$(u)(v)(t)[(u \text{ is the successor of } v \text{ in the ordering } <^{\gamma})$$
$$\rightarrow \pi(\langle 1, \langle u, t\rangle\rangle) = F[\lambda t\, \pi(\langle 1, \langle v, t\rangle\rangle)](t)] \;\&$$
$$(u)(s)(t)(u \text{ is a limit element in the ordering } <^{\gamma})$$
$$\rightarrow (\pi(\langle 1, \langle u, \langle s, t\rangle\rangle\rangle) = 0 \equiv \pi(\langle 1, \langle s, t\rangle\rangle) = 0 \;\&\; s <^{\gamma} u)].$$

We have[9] the following

COROLLARY. *If $<^{\gamma}$ is a well ordering and γ is in $\mathfrak{U}$, then $\mathfrak{H}$-completion π of γ is in $\mathfrak{U}$.*

REMARK. Analogously to our Corollary, we can define partial hierarchies of the Δ_2^1-functions. Let $<^{\gamma}$ be a well ordering. We can define a sequence θ_z^{γ} of representing functions of sets for z in the field of the relation $<^{\gamma}$ by the following condition:

1°. If z is the first element of $<^{\gamma}$, then θ_z^{γ} is identically zero.

2°. If z is the successor of y in $<^{\gamma}$, then θ_z^{γ} is equal to $F[\theta_y^{\gamma}]$.

3°. If z is a limit element in $<^{\gamma}$, then $\theta_z^{\gamma}(t)=0$ if and only if $(t)_1 <^{\gamma} z$ and $\theta_{(t)_1}^{\gamma}((t)_0)=0$.

If γ is a Δ_2^1-ordinal, then every θ_z^{γ} is a Δ_2^1-function and so θ_z^{γ} is a partial hierarchy of the Δ_2^1-functions which is necessarily incomplete. It might occur that the hyperdegree of γ is not reached by those of θ_z^{γ} for some γ, and this fact may prevent us from constructing a complete hierarchy for the Δ_2^1-functions from below.

I thank Professor Motokiti Kondô for his constant encouragement and valuable aid in the preparation of the paper.

REFERENCES

1. J. W. Addison, *Hierarchies and the axiom of constructibility*, Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, pp. 355–362.

2. ———, *Some consequences of the axiom of constructibility*, Fund. Math. 46 (1959), 337–357.

[9] From our proof of the Theorem 4, we can see that there is a recursive predicate S such that $F[\beta]$ is the unique solution of the predicate $\lambda\gamma(\alpha)(Ex)S(\gamma, \beta, \alpha, x)$ for every β.

3. J. W. Addison and S. C. Kleene, *A note on function quantification*, Proc. Amer. Math. Soc. **8** (1957), 1002–1006.

4. R. O. Gandy, *On a problem of Kleene's*, Bull. Amer. Math. Soc. **66** (1960), 501–502.

5. ———, *Complete predicates for the analytic hierarchy*, Abstract 150-62, Notices Amer. Math. Soc. **9** (1962), 222.

6. S. C. Kleene, *Introduction to metamathematics*, Van Nostrand, New York, 1952.

7. ———, *Arithmetical predicates and function quantifiers*, Trans. Amer. Math. Soc. **79** (1955), 312–340.

8. ———, *On the forms of the predicates in the theory of constructive ordinals* (second paper), Amer. J. Math. **77** (1955), 405–428.

9. ———, *Hierarchies of number-theoretic predicates*, Bull. Amer. Math. Soc. **61** (1955), 192–213.

10. L. Kantrovitch and E. Livenson, *Memoir on the analytical operations and projective sets* (I), Fund. Math. **18** (1932), 214–279.

11. M. Kondô, *Sur l'uniformisation des complêmentaires analytiques et les ensembles projectifs de la seconde classe*, Japan J. Math. **15** (1938), 197–230.

12. G. Kreisel, *Review of Shütte*, J. Symb. Logic **25** (1960), 243–249.

13. K. Kunugui, *Sur un thêorème d'existence dans la thêorie des ensembles projectifs*, Fund. Math. **29** (1937), 166–181.

14. N. Lusin, *Leçons sur les ensembles analytiques et leurs applications*, Gauthier-Villars, Paris, 1930.

15. W. Markwald, *Zur Theorie der konstruktiven Wohlordnungen*, Math. Ann. **127** (1954), 135–149.

16. J. Myhill, *Creative sets*, Z. Math. Logik Grundlagen Math. **1** (1955), 97–108.

17. E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.

18. Y. Sampei, *On the evaluation of the projective class of sets defined by transfinite induction*, Comment. Math. Univ. St. Paul **7** (1958), 21–26.

19. ———, *On the uniformization of the complement of an analytic set*, Comment. Math. Univ. St. Paul. **10** (1960), 57–62.

20. J. R. Shoenfield, *The problem of predicativity*, Essays on the Foundations of Mathematics, pp. 132–139, North-Holland, Amsterdam 1961.

21. ———, *The form of the negation of a predicate*, *Recursive function theory*, Amer. Math. Soc. (1962), 131–138.

22. C. Spector, *Recursive well-orderings*, J. Symb. Logic **20** (1955), 151–163.

23. ———, *Measure-theoretic construction of incomparable hyperdegrees*, J. Symb. Logic **23** (1958), 280–288.

24. T. Tugué, *Predicates recursive in a type-2 object and Kleene hierarchies*, Comment. Math. Univ. St. Paul. **8** (1960), 97–117.

25. H. Wang, *Alternative proof of a theorem of Kleene*, J. Symb. Logic **23** (1958), 250.

TOKYO METROPOLITAN UNIVERSITY, TOKYO, JAPAN