RESIDUAL NILPOTENCE AND RELATIONS IN FREE GROUPS

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Introduction. Relations between elements of a free group lead directly to identities for all groups, which are of considerable importance. In 1958 R. C. Lyndon [7] initiated a study of relations in free groups; in particular Lyndon proved that if g_1, g_2, g are elements of a free group and

$$g_1^2 g_2^2 = g^2,$$

then g_1, g_2, g generate a cyclic group. Since then a number of generalizations of this theorem have been obtained by E. Schenkman [10], John Stallings [12], and Gilbert Baumslag [1; 2]. The most recent result of this kind proved by M. P. Schützenberger [11] (cf. also R. C. Lyndon and M. P. Schützenberger [8]) and, independently, also by Arthur Steinberg [13], states that if

$$g_1^p g_2^q = g^r,$$

where now p, q and r are integers greater than 1, then again g_1, g_2, g generate a cyclic group.

Similarly, if instead

$$g_1^{-1} g_2^{-1} g_1 g_2 = g^r \qquad (r > 1),$$

then once more g_1, g_2, g generate a cyclic group (M. P. Schützenberger [11], Gilbert Baumslag [3], and A. Karass, W. Magnus and D. Solitar [6], and Arthur Steinberg [13]).

The purpose of this note is to announce the following theorem which contains both the aforementioned theorems as special cases.

THEOREM 1. *Let $w = w(x_1, x_2, \cdots, x_n)$ be an element of a free group F freely generated by $x_1, x_2, \cdots, x_n$ which is neither a proper power nor a primitive.*[2] *If $g_1, g_2, \cdots, g_n, g$ are elements of a free group connected by the relation*

$$w(g_1, g_2, \cdots, g_n) = g^m \qquad (m > 1),$$

then the rank of the group generated by $g_1, g_2, \cdots, g_n, g$ is at most $n-1$.

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[2] An element in a free group is termed primitive if it can be included in a set of free generators.

The authors would like to justify this joint announcement of work done independently by pointing out that Theorem 1 was obtained first by Baumslag although a more general result was already implicit in the Ph.D. thesis of Steinberg.

It is worth noting that the two proofs of Theorem 1 are completely different. On the one hand, Steinberg makes use of the Freiheitssatz of Wilhelm Magnus [9]. The information obtained in this way is so precise that a more general result than Theorem 1 can be obtained. Baumslag's proof, on the other hand, is less incisive, making use of residual properties and groups with unique roots (cf. [4; 5]). However, it seems likely that this approach might well be of value in treating analogous questions for other varieties of groups. Consequently, both proofs are of independent interest; they will appear separately, in detail, elsewhere.

References

1. Gilbert Baumslag, *On a problem of Lyndon*, J. London Math. Soc. **35** (1960), 30–32.

2. ———, *On generalised free products*, Math. Z. **78** (1962), 423–438.

3. ———, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303.

4. ———, *Some remarks on nilpotent groups with roots*, Proc. Amer. Math. Soc. **12** (1961), 262–267.

5. K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.

6. A. Karass, W. Magnus and D. Solitar, *Elements of finite order in groups with a single defining relation*, Comm. Pure Appl. Math. **13** (1960), 57–66.

7. R. C. Lyndon, *The equation* $a^2b^2=c^2$ *in free groups*, Michigan Math. J. **6** (1959), 89–95.

8. R. C. Lyndon and M. P. Schützenberger, *The equation* $a^m=b^nc^p$ *in a free group*, Michigan Math. J. 9 (1962), 289–298.

9. W. Magnus, *Ueber diskontinuierliche Gruppen mit einer definierenden Relation* (*Der Freiheitssatz*), J. Reine Angew. Math. **163** (1930), 141–165.

10. E. Schenkman, *The equation* $a^nb^n=c^n$ *in a free group*, Ann. of Math. (2) **70** (1959), 562–564.

11. M. P. Schützenberger, *Sur l'équation* $a^{2+n}=b^{2+m}c^{2+p}$ *dans un groupe libre*, C. R. Acad. Sci. Paris **248** (1959), 2435–2436.

12. John Stallings, *On certain relations in free groups*, Notices Amer. Math. Soc. **6** (1959), 532.

13. Arthur Steinberg, Ph.D. thesis, New York University, 1962.

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