A NOTE ON THE ADJOINT OF A PERTURBED OPERATOR

BY R. W. BEALS[1]

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Let X and Y be Banach spaces. By an *operator* from X to Y we mean a linear operator with domain $D(T)\subseteq X$ and range $R(T)\subseteq Y$. An operator T from X to Y is said to be a *Fredholm operator* if T is closed, the null space $N(T)$ has finite dimension, and the range $R(T)$ is closed and has finite codimension in Y. We denote by $\Phi(X, Y)$ the set of all Fredholm operators from X to Y. If $T\in\Phi(X, Y)$, the *index* of T is defined to be

$$\operatorname{ind}(T) = \dim(N(T)) - \operatorname{codim}(R(T)).$$

Suppose $T\in\Phi(X, Y)$. Since T is closed, the graph $G(T)$ is a closed subspace of $X\times Y$. An operator C from X to Y is said to be *T-compact* if C is closable, $D(C)\supseteq D(T)$, and the mapping $(x, Tx)\to Cx$ is compact as an operator from $G(T)$ into Y.

The following results are well known:

(1) If $T\in\Phi(X, Y)$ and C is T-compact, then $T+C\in\Phi(X, Y)$ and $\operatorname{ind}(T+C)=\operatorname{ind}(T)$. (See [**2**, Theorem 2.6].)

(2) If T is a closed operator from X to Y and $D(T)$ is dense in X, then T is in $\Phi(X, Y)$ if and only if the adjoint operator T^* is in $\Phi(X^*, Y^*)$. If so, then $\operatorname{ind}(T^*)=-\operatorname{ind}(T)$. (For by [**1**, Theorem A], $R(T)$ is closed if and only if $R(T^*)$ is closed. If so, then $\dim(N(T^*))=\operatorname{codim}(R(T))$ and $\operatorname{codim}(R(T^*))=\dim(N(T))$.)

LEMMA. *Suppose $S\in\Phi(X, Y)$, $T\in\Phi(X, Y)$, and $S\subseteq T$. Then $\operatorname{ind}(S)\leqq\operatorname{ind}(T)$, and equality holds if and only if $S=T$.*

PROOF. If $S\subseteq T$, then $N(S)\subseteq N(T)$ and $R(S)\subseteq R(T)$, so the inequality is obvious. Equality implies $N(S)=N(T)$ and $R(S)=R(T)$, so $S=T$.

PROPOSITION. *Suppose $T\in\Phi(X, Y)$ and $D(T)$ is dense in X. If C is an operator from X to Y such that C is T-compact and C^* is T^*-compact, then $T^*+C^*=(T+C)^*$.*

PROOF. From (1) and (2) it follows that $T^*+C^*\in\Phi(X^*, Y^*)$ and $(T+C)^*\in\Phi(X^*, Y^*)$. Furthermore

$$\begin{aligned}\operatorname{ind}(T^* + C^*) = \operatorname{ind}(T^*) &= -\operatorname{ind}(T)\\ &= -\operatorname{ind}(T + C) = \operatorname{ind}((T + C)^*).\end{aligned}$$

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[1] National Science Foundation Graduate Fellow.

It is obvious that $T^*+C^* \subseteq (T+C)^*$, and equality follows from the lemma.

REMARK. If C is T-compact, C^* need not be T^*-compact. In fact it can happen that $D(C^*) \cap D(T^*) = (0)$, and therefore $(T+C)^* \neq T^*+C^*$, even when T is the inverse of a positive definite compact Hermitian operator in a Hilbert space.

REFERENCES

1. F. E. Browder, *Functional analysis and partial differential equations*. II, Math. Ann. **145** (1962), 81–226.

2. I. C. Gohberg and M. G. Krein, *The basic propositions on defect numbers, root numbers, and indices of linear operators*, Uspehi Mat. Nauk **12** (1957), 43–118; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 185–264.

YALE UNIVERSITY