CLASSIFICATION OF OPERATORS BY MEANS OF THE OPERATIONAL CALCULUS[1]

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1. **Introduction.** Let $\boldsymbol{A}=\boldsymbol{A}(\Delta)$ be a topological algebra of complex valued functions defined on a subset Δ of the complex plane, with the usual operations. Suppose that $\boldsymbol{A}$ contains the restrictions to Δ of polynomials. Let $B(X)$ be the Banach algebra of all bounded linear operators on the Banach space X into itself. We say that an operator T is *of class* $\boldsymbol{A}$ (notation: $T\in(\boldsymbol{A})$) if there exists a continuous representation $f\rightarrow T(f)$ of $\boldsymbol{A}$ into $B(X)$ such that $T(1)=I$ and $T(z)=T$. Such a representation is called an $\boldsymbol{A}$-*operational calculus* for T. A class $(\boldsymbol{A})$ may be as wide as $B(X)$ (if $\boldsymbol{A}$ consists of all entire functions with the topology of uniform convergence on every compact), or as narrow as the class of hermitian operators with spectrum in a given compact Δ (if $\boldsymbol{A}=C(\Delta)$, $T(\cdot)$ is norm decreasing, and X is a Hilbert space). Related approaches are found in [**3**; **5**].

2. **Restrictions on** $\boldsymbol{A}$. Let $H(\Delta)$ denote the algebra of all complex valued functions which are locally holomorphic in a neighborhood of Δ, with the usual topology.

CONDITION 1. If $f\in H(\Omega)$ for a compact $\Omega\neq\emptyset$, then there exists $f_0\in\boldsymbol{A}(\Delta)$ such that $f_0=f$ on $\Delta\cap\Omega_0$, for some neighborhood Ω_0 of Ω.

This condition excludes in particular the noninteresting case $\boldsymbol{A}(\Delta)=H(\Delta)$. *We shall consider here only* $\Delta=\boldsymbol{R}$ (the real line) or $\Delta=\boldsymbol{C}$ (the complex plane), and assume that $\boldsymbol{A}_0=\{f\in\boldsymbol{A}\mid f$ has compact support$\}$ is dense in $\boldsymbol{A}$.

Fix $f\in\boldsymbol{A}_0$. If $g\in H(\operatorname{Spt} f)$, Condition 1 implies the existence of $g_0\in\boldsymbol{A}$ such that $g_0=g$ on $\operatorname{Spt} f$. The map $M_f\colon H(\operatorname{Spt} f)\rightarrow\boldsymbol{A}$ given by $M_f g=fg_0$ is well defined.

CONDITION 2. The map $M_f\colon H(\operatorname{Spt} f)\rightarrow\boldsymbol{A}$ is continuous, for each $f\in\boldsymbol{A}_0$. A topological algebra $\boldsymbol{A}$ as in §1 which satisfies also Conditions 1 and 2 is called a *basic algebra* (compare [**5**]). Example: C^n for $0\leqq n\leqq\infty$.

3. **Restrictions on** $T(\cdot)$.

CONDITION 3. $T(\cdot)$ *has compact support* (denoted by Σ). If $g\in H(\Sigma)$ and $g_0\in\boldsymbol{A}$ is such that $g_0=g$ in a neighborhood of Σ (cf. Condition 1),

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define a representation $T_H: H(\Sigma) \to B(X)$ by $T_H(g) = T(g_0)$. T_H is well defined. We call it the *restriction of* $T(\cdot)$ *to* $H(\Sigma)$.

CONDITION 4. $T_H: H(\Sigma) \to B(X)$ *is continuous.*

DEFINITION. An *operational calculus* (o.c.) is the object $(\boldsymbol{A}, T(\cdot))$ consisting of a *basic algebra* $\boldsymbol{A}$ and of a *continuous representation* $T(\cdot)$ of $\boldsymbol{A}$ in $B(X)$ which satisfies Conditions 3 and 4 as well as the normalizing condition $T(1) = I$. An operator T is *of class* $\boldsymbol{A}$ ($T \in (\boldsymbol{A})$) if there exists an o.c. $(\boldsymbol{A}, T(\cdot))$ such that $T(z) = T$.

4. **Basic facts.** Let $T \in (\boldsymbol{A})$ and let $(\boldsymbol{A}, T(\cdot))$ be an o.c. for T. Then

1. $\Sigma = \sigma(T)$ (the spectrum of T).
2. The restriction of $T(\cdot)$ to $H(\Sigma)$ is the usual analytic operational calculus for T.

Property 1 motivates the convention: for real operators (i.e., $\sigma(T) \subset \boldsymbol{R}$), we take $\Delta = \boldsymbol{R}$.

It is reasonable to require that if $T \in (\boldsymbol{A})$, also $tT \in (\boldsymbol{A})$ for $t \in \boldsymbol{R}$. This corresponds to the following requirement on $\boldsymbol{A}$: if $f \in \boldsymbol{A}$, then $f_t \in \boldsymbol{A}$ (where $f_t(x) = f(tx)$) and the map $f \to f_t$ of $\boldsymbol{A}$ into itself is continuous ($t \in \boldsymbol{R}$). If $\boldsymbol{A}$ has this property, we say that $\boldsymbol{A}$ is *homogeneous.*

THEOREM 1 ("CLASSIFICATION THEOREM"). *If T is a real operator of class $\boldsymbol{A}$ for a homogeneous Banach algebra $\boldsymbol{A}$, then T is of class C^n for some $n < \infty$.*

5. **Operators of finite class.** We say that T is *of finite class* if it is of class C^n for some $n < \infty$ (cf. Theorem 1).

The proof of the Classification Theorem is based on the following characterization of operators of finite class.

THEOREM 2. *T is a real operator of finite class if and only if* $\|e^{itT}\| = O(|t|^k)$ $(t \in \boldsymbol{R}, |t| \to \infty)$ *for some* $0 \leq k < \infty$.

More precisely, if T is real of class C^n, then $\|e^{itT}\| = O(|t|^n)$; conversely, the latter condition is sufficient for T to be real of class C^{n+2}.

6. **Relationship with spectral operators.** Theorem 2 implies in particular that sums and products of commuting real operators of finite class are of finite class. Spectral operators of finite type (cf. [2]) are operators of finite class. The converse is false by the preceding remark, even in reflexive Banach spaces (cf. [6, pp. 303–304]). However, it is true that $T \in (C)$ if and only if T is spectral of scalar type (if X is weakly complete). In particular, when X is a Hilbert space, (C) is the class of all operators which are similar to normal operators.

Moreover, T is normal if and only if it is of class C and has a norm-decreasing C—o.c.

If $T\in(C^n)$ $(1\leqq n<\infty)$, $x\in X$ and $x^*\in X^*$, then $x^*T(\cdot)x$ is a continuous linear functional on C^n with compact support Σ; as such, it has representations of the form $\sum_{0\leqq j\leqq n}\mu_j^{(j)}$, where μ_j are regular finite Borel measures with supports in an arbitrary neighborhood of Σ. We say that T is singular if $x^*T(\cdot)x$ has such a representation in which μ_j $(j\geqq 1)$ are singular with respect to Lebesgue measure (for all x, x^*).

THEOREM 3. *A real operator on a reflexive Banach space is singular of class C^n $(n\geqq 1)$ if and only if it is spectral of type n and its nilpotent part N and resolution of the identity $E(\cdot)$ are such that $x^*NE(\cdot)x$ is singular with respect to Lebesgue measure for all x and x^*.*

In other words, singular real operators of class C^n have a "Jordan canonical form" $T=S+N$, where S is scalar and real, $N^{n+1}=O$, and S commutes with N (when X is reflexive).

7. **Characterizations of** (C^n). Theorem 2 gives a simple characterization of $\bigcup_{n\geqq 0}(C^n)$ in terms of a growth condition on the group e^{itT}, $t\in R$. In order to characterize in a similar way a *given* class (C^n), we need some "averages" of e^{itT}. Let $L_{1,n}=\{f\in L_1(R)\mid t^jf(t)\in L_1(R);\ 0\leqq j\leqq n\}$; if $f\in L_{1,n}$, its Fourier transform $\hat{f}$ is obviously in C^n. For $g\in C^n$ and Δ compact, write

$$\|g\|_{n,\Delta}=\sum_{0\leqq j\leqq n}\frac{1}{j!}\sup_\Delta|g^{(j)}|.$$

DEFINITION. Let n be a non-negative integer, Δ a compact interval, and $T\in B(X)$. *The nth variation of T over Δ* is defined by

$$v_n(T;\Delta)=\sup\left\|\int f(t)e^{itT}dt\right\|,$$

where the sup is taken over all $f\in L_{1,n}$ with $\|\hat{f}\|_{n,\Delta}=1$.[2]

In general, $v_n(T;\Delta)=\infty$. However, we have

THEOREM 4. *T is real of class C^n if and only if $v_n(T;\Delta)<\infty$ for some compact interval Δ. (In this case, $\sigma(T)\subseteq\Delta$.)*

This generalizes Theorem 6 in [4]. As a first corollary, we have the following generalization of results of Bade's [1]:

[2] If the integral does not converge in the uniform operator topology for some f in $L_{1,n}$, we set its norm equal to $+\infty$.

THEOREM 5. *Let $T_a \in B(X)$ be a net converging to $T \in B(X)$ in the strong operator topology. Suppose that, for some n and some compact interval Δ, $\sup_a v_n(T_a; \Delta) < \infty$. Then T (as well as all T_a) is of class C^n with spectrum in Δ, and $T(f) = \lim T_a(f)$ ($f \in C^n$) in the strong operator topology.*

Applying Theorem 5, we get

THEOREM 6. *Let T and S be two commuting real operators in Hilbert space, $T \in (C^n)$ and $S \in (C)$. Then $T+S \in (C^n)$ and*

$$(T + S)(f) = \int T(f_t)dE(t), \qquad f \in C^n,$$

where $E(\cdot)$ is the resolution of the identity for S, $f_t(x) = f(t+x)$ and the integral exists in the strong operator topology.

An analogous result holds in arbitrary Banach spaces, but it would be too long to state it here. Theorem 6 generalizes known results about spectral operators.

The growth condition in Theorem 4 may be expressed in terms of the resolvent. For example, for $n=0$, we get: A real operator is of class C if and only if the integral $\int |x^*[T(t-is; T) - R(t+is; T)]x| dt$ is uniformly bounded when $s \to 0+$, for all unit vectors x and x^*. In this case, the C-o.c. for T is given by

$$T(f) = \lim_{s \to 0+} \frac{1}{2\pi i} \int f(t)[R(t - is; T) - R(t + is; T)]dt, \qquad f \in C.$$

This explicit representation of the o.c. is well known for hermitian operators (compare [7]). Another explicit representation of the C^n-o.c., together with a characterization of (C^n), may be obtained as follows. For $u \geqq 0$, $t \in \mathbf{R}$, $m = 1, 2, \cdots$, and $T \in B(X)$ arbitrary, let

$$G_m(t, u) = \frac{1}{2\pi} \int \exp - [(v/m)^2 + u|v| + ivt]e^{-ivT}dv,$$

and

$$T_m(f; u) = \int f(t)G_m(t, u)dt, \qquad f \in C_0^n,$$

where $C_0^n = \{f \in C^n | f \text{ has compact support}\}$.

THEOREM 7. *A real operator T is of class C^n if and only if, for every $f \in C_0^n$, $T_m(f; u) \to T(f; u) \in B(X)$ in the weak operator topology ($m \to \infty$),* *uniformly with respect to* u $(u \geqq 0)$, *and, for some constant* $M > 0$ *and some compact interval* Δ, $\|T(f;\, u)\| \leqq M\|f\|_{n,\Delta}$ $(u \geqq 0)$.

In this case, the C^n — o.c. for T is given by $T(f) = T(f_0;\, 0)$, $f \in C^n$, where $f_0 \in C_0^n$ is such that $f_0 = f$ on Δ.

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