THE GEOMETRY OF IMMERSIONS. II

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This announcement is the sequel to the announcement *Geometry of immersions.* I, which appeared in the September 1963 issue of this journal. In the present announcement we develop the characteristic class theory of Thom (see [6], [7]) to include some situations of differential geometric interest. We then apply these techniques of differential topology to some problems of extrinsic differential geometry. Most notable among them is §6 which deals with the problem of counting umbilic points on an immersed hypersurface.

In this announcement we draw freely on the definitions contained in our first announcement. Full details will appear in a separate publication.

1. G-structures on vector bundles. Their singularities and characteristic classes.

DEFINITION 1.1. Let $\phi_1: G \to \mathrm{GL}(p, R)$ and $\phi_2: H \to \mathrm{GL}(q, R)$ be faithful smooth representations of the Lie groups G and H in $\mathrm{GL}(p, R)$ and $\mathrm{GL}(q, R)$ respectively. This induces in a natural way an action of $G \times H$ on $\mathrm{Hom}(R^p, R^q)$. Let K be a regular manifold collection of submanifolds of $\mathrm{Hom}(R^p, R^q)$. Then K is called a *model $G \times H$ singularity with respect to ϕ_1 and ϕ_2*, if K is invariant under the action of $G \times H$ on $\mathrm{Hom}(R^p, R^q)$.

DEFINITION 1.2. Let $\xi = (\pi: E \to X)$ be a smooth vector bundle of fiber dimension q. Let $\pi_E: P_E \to X$ be the principal $\mathrm{GL}(q, R)$ bundle associated to E. Let G be a Lie group. Let $\phi: G \to \mathrm{GL}(q, R)$ be a faithful representation of G. A *(G, ϕ)-structure on E* is given by a principal G-bundle $\pi_G: P_G \to X$, and a smooth mapping $\tilde{\phi}: P_G \to P_E$ such that

(a) $\pi_E \cdot \tilde{\phi} = \pi_G$,

(b) $\tilde{\phi}$ is smooth and 1-1,

(c) for each $u \in P_G$ and $g \in G$, $\tilde{\phi}(u)\phi(g) = \tilde{\phi}(ug)$.

REMARK. A *(G, ϕ)-structure* on a vector bundle $\xi = (\pi: E \to X)$ of fiber dimension q, gives rise to a reduction of the structural group $\mathrm{GL}(q, R)$ to $\phi(G)$. Hence the representation ϕ allows us to consider the vector bundle ξ as a vector bundle associated with the principal G-bundle P_G, because the G action on the fibers will be linear.

DEFINITION 1.3. Let $\pi: E \to X$ and $\pi': E' \to Y$ be smooth vector bundles of fiber dimension p and q respectively. Let there be given a (G, ϕ_1)-structure on E and an (H, ϕ_2)-structure on E'. Let K be a model $G \times H$ singularity with respect to the representations, ϕ_1 and ϕ_2. Let $\mathrm{Hom}(E, E')$ denote the vector bundle over $X \times Y$ defined in §2 of [2]. Then the (G, ϕ_1)-structure on E and the (H, ϕ_2)-structure on E' induce a $(G \times H, \phi_1^{tr^{-1}} \times \phi_2)$-structure on $\mathrm{Hom}(E, E')$. Hence we can consider $\mathrm{Hom}(E, E')$ as a vector bundle with fiber $\mathrm{Hom}(R^p, R^q)$, structural group $G \times H$, and base $X \times Y$. Hence K defines a regular manifold collection $K(X, Y)$, of submanifolds of $\mathrm{Hom}(E, E')$. Let $K = K_1 \cup \cdots \cup K_s$, then $K(X, Y) = K_1(X, Y) \cup \cdots \cup K_s(X, Y)$, and the $\mathrm{codim}(K_i) = \mathrm{codim}(K_i(X, Y))$. $K(X, Y)$ is called the $G \times H$ *singularity defined by the* (G, ϕ_1)-*structure on* E, *the* (H, ϕ_2)-*structure on* E', *and the model* $G \times H$ *singularity* K.

DEFINITION 1.4. *Let* $\pi: E \to X$ *and* $\pi': E \to Y$ *be smooth vector bundles.* Let there be given a (G, ϕ_1)-structure on E and an (H, ϕ_2)-structure on E'. Let K be a model $G \times H$ singularity, and let $K(X, Y)$ be the $G \times H$ singularity defined by the G-structure on E the H-structure on E', and the $G \times H$ singularity K. Let $f: E \to E'$ be a smooth vector bundle homomorphism covering $\bar{f}: X \to Y$. This induces a map $\hat{f}: X \to \mathrm{Hom}(E, E')$ defined by $\hat{f}(x) = f \mid E_x \in \mathrm{Hom}(E_x, E'_{\bar{f}(x)})$, for $x \in X$. We say x is a K-*singular point of* f if $\hat{f}(x) \in K(X, Y)$. We say that f *displays the singularity transversally* if $\hat{f}(X)$ meets $K(X, Y)$ transversally. The set $\hat{f}^{-1}(K(X, Y)) = S_K(f)$ is called the set of K-*singular points of* f. If f displays K transversally then $S_K(f)$ is a manifold collection of the same codimension as $K(X, Y)$.

Let the coefficient ring for all characteristic classes be Z_2.

DEFINITION 1.5. Let $\pi: E \to X$ be a smooth vector bundle. Let $\pi_G: P_G \to X$ be the principal G-bundle given by a (G, ϕ)-structure on E. The *characteristic classes of the G-structure* (G, ϕ) on E are defined to be the characteristic classes of P_G. If the maximal compact subgroup G_c of G is the orthogonal group $O(k)$, then the characteristic classes of the G-structure are just the Stiefel-Whitney classes of P_G. These will be called the *Stiefel-Whitney classes associated with the* (G, ϕ)-*structure on* E. We will denote these classes by $W_i^G(E) \in H^i(X)$.

With these definitions we get the following generalizations of the theorem of Thom (see [6], [7]).

THEOREM 1.6. (1) *Let* $\xi = (\pi: E \to X)$ *and* $\eta = (\pi': E' \to Y)$ *be smooth vector bundles of fiber dimension* p *and* q *respectively. Let* ξ *be given a* (G, ϕ_1)-*structure and* η *an* (H, ϕ_2)-*structure. Assume the maximal compact subgroups* G_c *and* H_c *are respectively* $O(k)$ *and* $O(t)$.

(2) *Let* $K = K_1 \cup \cdots \cup K_a$ *be a model* $G \times H$ *singularity of co-dimension s. Assume that K carries a fundamental class, that K is closed in* $\mathrm{Hom}(R^p, R^q)$, *and that K_1 is a regular submanifold. (These are all satisfied if K is a real algebraic variety.)*

(3) *Let* $f\colon E \to E'$ *be a smooth homomorphism of vector bundles covering $\bar{f}\colon X \to Y$. Assume that f displays the singularity K transversally.*

Then $S_K(f)$, the manifold collection of K singular points of f carries a fundamental class $[S_K(f)]$. Let $i\colon S_K(f) \to X$ be the inclusion map and let D_X be Poincaré duality in X. There exists a polynomial P, which is independent of f, such that

$$D_X i_*[S_K(f)] = P(W_j^G(E), \bar{f}^* W_j^H(E')).$$

COROLLARY 1.7. *If the P_H of the H-structure in 1.6 is isomorphic to the trivial bundle, then the class $[S_K(f)]$ is independent of $\bar{f}$.*

REMARK. (1) An exact analogue of 1.6 can be proven for arbitrary G- and H-structures. However it is the less general theorem above that we apply in what follows.

(2) I. Porteous (see [1]) has calculated the polynomials of Theorem 1.6 in the following case. Let $n = \min(p, q)$, let $G = \mathrm{GL}(p, R)$, let $H = \mathrm{GL}(q, R)$, let $\phi_1 = id_G$ and let $\phi_2 = id_H$. Let us denote by Z_a the model $G \times H$ singularity defined by

$$Z_a = (\sigma \in \mathrm{Hom}(R^p, R^q) \mid \mathrm{rank}\ \sigma = n - a).$$

Then $\bar{Z}_a = \bigcup_{a \geq a'} Z_{a'}$. $\bar{Z}_a$ is a real algebraic variety and therefore satisfies all the hypotheses of 1.6. Porteous calculates the universally defined polynomial for the singularity $Z_a(X, Y)$ under a slightly different transversality assumption. It is not hard to show that the transversality assumption of 1.6 is stronger than Porteous', and hence his formulas hold.

2. **Applications to osculating maps.** Let us preserve the notation of 1.6. Assume that dim $X = n$ and that dim $Y = N$. Let $(D^{(k)}, k = 1, 2, \cdots)$ be a sequence of kth order symmetric linear connections on Y. Let us assume that $E = T_p(X)$ and that $E' = T_1(Y)$. Let $G = J^p(n)$, the group of invertible p-jets of maps from R^n into R^n which send the origin into the origin. The differential structure on X endows $T_p(X)$ with a canonical $J^p(n)$-structure. Let there be given some arbitrary H-structure on $T_1(Y)$. Let K be a model $J^p(n) \times H$ singularity with respect to the given structures. Let $f\colon X \to Y$ be a smooth map. The vector bundle homomorphism that we are interested in studying is the pth order osculating map $\nabla_p T_p(f)\colon T_p(X) \to T_1(Y)$.

The following proposition shows that condition 3 of Theorem 1.6 is not very restrictive in the case outlined above.

PROPOSITION 2.1. *Let X and Y be smooth manifolds of dimension n and N respectively. Let $(D^{(k)}, k=1, 2 \cdots)$ be a sequence of kth order symmetric linear connections on Y. Let there be given an H-structure on $T_1(Y)$ and a G-structure on $T_p(X)$. Let K be any model $G \times H$ singularity, and let $K(X,Y)$ be the corresponding singularity induced by the G-structure, the H-structure and K. Then the set of $f \in C(X, Y)$ such that $(\nabla_p T_p(f))^{\wedge}(X)$ meets $K(X,Y)$ transversally is dense in $C(X, Y)$.*

The remainder of this announcement will deal with the applications of 2.1 and 1.6 to some concrete problems of "affine" and "Riemannian" singularities.

3. **Inflection points.** We will fix the following notation for the next two sections. Let X and Y be smooth manifolds of dimension n and N respectively. Let $(D^{(k)}, k=1, 2 \cdots)$ be a sequence of kth order symmetric linear connections on Y. Let $\nu(n, p)$ denote the fiber dimension of $T_p(X)$. Assume that $N > \nu(n, p)$.

DEFINITION 3.1. Let $f: X \rightarrow Y$ be a pth order nondegenerate immersion of X in Y, and let $N_{p,f,\Upsilon}(X)$ be the pth order normal bundle of X in Y with respect to f.

Let $\nu_{p,f}: O^{p+1}T_1(X) \rightarrow N_{p,f,\Upsilon}(X)$ be the pth order normal form of f. $x \in X$ is said to be a *pth order inflection point of f* if $\nu_{p,f}$ is not of maximal rank on $O^{p+1}T_1(X)_x$.

Let $I^p(X, Y)$ denote the space of pth order nondegenerate immersions of X in Y. Let $S_i(\nu_{p,f}) = (x \in X \mid \text{rank } (\nu_{p,f})$ on

$$O^{p+1}T_1(X)_x = \min(\nu(n, p+1) - \nu(n, p), N - \nu(n, p)) - i).$$

THEOREM 3.2. (a) *There exists an open dense subset $D \subseteq I^p(X, Y)$ such that if $f \in D$ then the set of pth order inflection points of f is a manifold collection $I_f = S_1(\nu_{p,f}) \cup \cdots \cup S_k(\nu_{p,f})$ of submanifolds of X. Furthermore we have the following results.*

(b) *Let $N \geq \nu(n, p+1)+n$ or let $\nu(n, p) < N \leq \nu(n, p+1)-n$. If $f \in D$ then $I_f = \varnothing$ and hence f is a $p+1$ order nondegenerate immersion.*

(c) *Let $\nu(n, p+1)-n < N < \nu(n, p+1)+n$. Let $f \in D$. $[S_i(\nu_{f,p})]^- = S_i(\nu_{p,f}) \cup \cdots \cup S_k(\nu_{p,f})$. Then $[S_i(\nu_{p,f})]^-$ carries a fundamental class $[S_i(\nu_{p,f})]^-$. Let D_X be Poincaré duality in X and let $j: S_i[(\nu_{p;f})]^- \subseteq X$ be the inclusion map. Then*

$$D_x j_* [S_i(\nu_{p,f})]^- = P(W_a(X), f^* W_b(Y)),$$

where P is a universally defined polynomial whose definition is inde-

*pendent of f, and where $W_a(X)$ and $W_b(Y)$ are the Stiefel-Whitney
classes of X and Y respectively.*

REMARK. The polynomial P can be computed from Porteous' for-
mulas, Whitney duality, and the formulas for the Whitney classes of
the tensor product of vector bundles.

The following examples are easy consequences of the above theo-
rems.

PROPOSITION 3.3. *Let S^1 be the circle. Let Y be unorientable. Let
$f \in I^{N-1}(S^1, Y)$. Assume that $f^*W_1(Y) \neq 0$. Then f has an $(N-1)$st
order inflection point. If $N = 2$ this tells us that any immersion f of the
circle in an unorientable two manifold has an inflection point if
$f^*W_1(Y) \neq 0$.*

PROPOSITION 3.4. *Let $\dim X = 2$, and let us be concerned with first
order inflection points. We will endow the target manifold Y with an
arbitrary symmetric connection.*

(a) *Let $X = P_2(R)$ and let $Y = P_6(R)$. Let $f: P_2(R) \to P_6(R)$ be the
natural embedding of P_2 as a projective plane. Then any immersion of
$P_2(R) \to P_6(R)$ homotopic to f has inflection points. The same fact is
true for the natural embedding of P_2 in P_4.*

PROPOSITION 3.5. *Let $X = P_2(C)$. We will again be concerned with
first order inflection points. We again endow the target manifold Y with
an arbitrary symmetric linear connection.*

(a) *If $Y = R^{17}$ (in fact any 17 manifold Y with $W_1(Y) = 0$ will
suffice), then any immersion of $P_2(C)$ in Y must have inflection points.*

(b) *If $Y = R^{14}$, then for any immersion of $P_2(C)$ in R^{14}, there exists
a point $x \in P_2(C)$ such that $\dim(\ker(\nu_{1,f}(x)) \geq 2$.*

Many more examples of this type can be given so we will stop here.

4. Totally geodesic points.

DEFINITION 4.1. Let $N > \nu(n, p)$, and let $f: X \to Y$ be a pth order
nondegenerate immersion. Let $N_{p,f,Y}(X)$ be the pth order normal
bundle of X in Y with respect to f, and let $\nu_{p,f}: O^{p+1}T_1(X) \to N_{p,f,Y}(X)$
be the pth order normal form. $x \in X$ is said to be a *pth order totally
geodesic point with respect to the connections of the immersion f*, if
$\nu_{p,f}$ is the zero map on the fiber $O^{p+1}T_1(X)_x$. If this true at all the
$x \in X$, f is called a *pth order totally geodesic immersion.*

REMARK. In the classical case $p = 1$ this definition is well known to
be equivalent to the usual definition.

PROPOSITION 4.2. *Let $N > \nu(n, p)$, and let $\dim X \geq 2$. Let $I^p(X, Y)$*

be the set of pth order nondegenerate immersions of X in Y. Then the set of $f \in I^p(X, Y)$ without pth order totally geodesic points is open and dense in $I^p(X, Y)$.

5. Parabolic points.

We will adopt the following notation for the next two sections. X will denote an n-dimensional manifold. Let $Y = R^{n+1}$ and endow Y with its usual Riemann (Euclidean) metric $\langle \ , \ \rangle$. Let $I(X, R^{n+1})$ denote the space of immersions of X in R^{n+1}. If $f \in I(X, R^{n+1})$ denote by $w_2 T_2(f): T_2(X) \to T_1(R^{n+1})$ the second order osculating map of f with respect to the usual connection on R^{n+1}. Denote by $(w_2 T_2)\hat{\ }(f): X \to \mathrm{Hom}(T_2(X), T_1(Y))$ the section of $\mathrm{Hom}(T_2(X), T_1(R^{n+1}))$ over the graph of f, determined by the homomorphism $w_2 T_2(f)$.

DEFINITION 5.1. Let $f \in I(X, R^{n+1})$. Let $N_f(X)$ and $\nu_f: O^2 T_1(X) \to N_f(X)$ be the normal bundle and the normal form of the immersion f respectively. Let $x \in X$ and U be a neighborhood of x in which we can define a unit normal vector field N to $f(U)$. For every point $x' \in U$ we can define a self adjoint linear transformation $S_{N(x')}: T_1(X)_{x'} \to T_1(X)_{x'}$ defined by the formula

$$\langle S_{N(x')} u, v \rangle = \langle \nu_f(u, v), N(x') \rangle, \qquad u, v \in T_1(X)_{x'}.$$

Let $K_{N,f}(x') = \mathrm{Det}(S_{N(x')})$. $K_{N,f}(x')$ is called the *Gauss-Kronecker curvature of f at x' with respect to the normal direction $N(x')$.*

DEFINITION 5.2. $x' \in X$ is called a *parabolic point* of the immersion f if $K_{N,f}(x') = 0$. This is clearly independent of the unit normal chosen, and depends only upon the immersion f and the Riemann structure on R^{n+1}.

PROPOSITION 5.3. *Let $f \in I(X, R^{n+1})$. Let $G_N(f): X \to P_n(R)$ be the Gauss normal map of f. x is a parabolic point of f if and only if x is not a regular point of $G_N(f)$.*

THEOREM 5.4. *There exists a model $J^2(n) \times O(n+1)$ singularity $P \subset \mathrm{Hom}(R^{\nu(n, \ p)}, R^{n+1})$ with the following properties.*

(a) *P is a real algebraic variety of codimension 1.*

(b) *Let $P(X, R^{n+1})$ be the singularity induced by the canonical $J^2(n)$-structure on $T_2(X)$, the Riemann structure on R^{n+1}, and the model singularity P. Then $x \in X$ is a parabolic point if and only if $(w_2 T_2)\hat{\ }(f)(x) \in P(X, R^{n+1})$.*

(c) *Let $D_p = (f \in I(X, R^{n+1}) | (w_2 T_2)\hat{\ }(f)$ take the singularity $P(X, R^{n+1})$ transversally). The elements of D_p are called parabolic generic immersions. The parabolic generic immersions are open and dense in $I(X, R^{n+1})$.*

THEOREM 5.5. *Let* $f \in I(X, R^{n+1})$. *Let* $S_p(f)$ *be the set of parabolic points of* f, *let* $i: S_p(f) \to X$ *be the inclusion map, and let* D_X *be Poincaré duality in* X. *If* $f \in D_q$ *then* $S_p(f)$ *is a manifold collection and carries a fundamental class* $[S_p(f)]$. *Furthermore we have the formula,* $D_X i_* [S_P(f)]$ $= p(n) W_1(X)$, *where* $p(n)$ *is a* Z_2 *valued function depending on the dimension* n *of* X *alone.*

5.5 and a deeper analysis of the singular variety yield the following corollaries.

COROLLARY 5.6. (a) *Let* X *be a compact orientable* n-*manifold. Let* $f \in D_P$. *Then the set of parabolic points of* f *is homologous to zero.*

(b) *Let* X *be a compact* 2-*manifold. Let* $f \in D_P$. *Then the set of parabolic points* $S_P(f)$ *is a set of disjoint circles which when considered as a homology class are homologous to zero.*

(c) *Let* X *be an odd dimensional, compact, unorientable manifold. Then any immersion of* X *in* R^{n+1} *has parabolic points.*

6. Umbilic points.

DEFINITION 6.1. Let $f \in I(X, R^{n+1})$. Let $x \in X$ and let U be a sufficiently small neighborhood of x so that we can define a unit vector field N normal to $f(U)$. Let $S_{N(x)}: T_1(X)_x \to T_1(X)_x$ be the self adjoint linear transformation defined in 5.1. The eigenvalues $k_i(x)$ of this operator are called the *principal curvatures of* f *at* x. These numbers are uniquely determined up to sign in the sense that if we picked $-N$ instead of N we would merely change the sign of $k_i(x)$.

DEFINITION 6.2. Let $f \in I(X, R^{n+1})$. A point $x \in X$ is called an *umbilic point of the immersion* f if all the principal curvatures of f at x are equal. It is clear that the property of being an umbilic point is independent of the unit normal chosen.

THEOREM 6.3. *There exists a model* $J^2(n) \times O(n+1)$ *singularity* $U \subset \operatorname{Hom}(R^{\nu(n, 2)}, R^{n+1})$ *with the following properties.*

(a) *U is a real algebraic variety of codimension*

$$\frac{n(n+1)}{2} - 1.$$

(b) *Let* $U(X, R^{n+1})$ *be the singularity induced by the* $J^2(n)$-*structure on* $T_2(x)$ *the Riemann structure on* R^{n+1}, *and the model singularity* U. *Then* $x \in X$ *is an umbilic point if and only if* $(w_2 T_2)^\wedge (f) \in U(X, R^{n+1})$.

THEOREM 6.4. *Let* $\dim X \geqq 3$. *Then the set of* $f \in I(X, R^{n+1})$ *such that* f *has no umbilic points is open and dense in* $I(X, R^{n+1})$.

THEOREM 6.5. *Let X be a compact orientable 2-manifold. Then there exists an open dense subset $D_U \subseteq I(X, R^3)$ with the following properties. If $f \in D_U$ then*

(a) *the set of umbilic points $S_U(f)$ is a finite set of points, and*

(b) *(the number of umbilic points) mod $2 = 0$.*

COROLLARY 6.6. *Let X be a compact orientable 2-manifold. Assume that the Euler characteristic $\chi(X) \neq 0$. Let $D_U \subset I(X, R^3)$ be the open dense subset of 6.5. If $f \in D_U$, then f has at least two umbilic points.*

REMARK. This last corollary gives a partial answer and generalization to a conjecture of C. Carathéodory. Carathéodory conjectured that any convex embedding of S^2 in R^3 has at least two umbilic points. In the case where the embedding is real analytic the conjecture has been settled in the affirmative, however the C^k case ($k \leq \infty$) is still an open problem.

Added in proof. Theorem 6.5 can be strengthened to the following theorem.

THEOREM 6.7. *Let X be a compact 2-manifold. Then there exists an open dense subset $D_U \subseteq I(X, R^3)$ with the following properties. If $f \in D_U$ then,*

(i) *the set of umbilic points of f is a finite subset of X, and*

(ii) *the index of each of these umbilic points is $\pm \frac{1}{2}$.*

Hence we have that (the number of umbilic points of $f) \geq 2|\chi(X)|$, where $\chi(X)$ is the Euler characteristic.

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