ON THE LIFTING PROPERTY AND DISINTEGRATION OF MEASURES[1]

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In what follows we shall present certain results concerning the lifting property of a space $L^\infty(Z, \mu)$, where Z is locally compact and μ a positive measure on Z. We shall discuss several applications and state several problems. Most of the results which will be presented here were jointly obtained by A. Ionescu Tulcea and the author, during the last few years. Some of these results have already appeared in print while others have not yet been published. The presentation will be divided as follows:

1. *The existence of a lifting.*
2. *The strong lifting.*
3. *Liftings commuting with certain groups of mappings.*
4. *Lower densities and liftings.*
5. *Integral representations of operators.*
6. *Disintegration of measures.*
7. *Point realizations of endomorphisms of $L^\infty(Z, \mu)$ spaces.*
8. *Final remarks.*

Notations. Here are some of the notations which will be used below. For a locally compact space X denote by $M(X)$ the vector space of all (Radon) measures on X and by $M_+(X)$ the cone of all positive measures on X. For every $\beta \in M_+(X)$ denote by $M^\infty(X, \beta)$ the algebra of all $f: X \to R$ which are bounded and β-measurable and by $N^\infty(X, \beta)$ the ideal of all $f \in M^\infty(X, \beta)$ which are locally β-negligible. By $L^\infty(X, \beta)$ we shall denote the quotient algebra $M^\infty(X, \beta)/N^\infty(X, \beta)$ and by $\pi: f \to \tilde{f}$ the canonical mapping of $M^\infty(X, \beta)$ onto $L^\infty(X, \beta)$. For $f \in M^\infty(X, \beta)$ and $g \in M^\infty(X, \beta)$ we write $f \equiv g$ whenever $\tilde{f} = \tilde{g}$ (therefore $f \equiv g$ means that f and g coincide locally almost everywhere, while $f = g$ means that f and g coincide everywhere). We denote by $C^\infty(X)$ the subalgebra of $M^\infty(X, \beta)$ consisting of all continuous functions $f \in M^\infty(X, \beta)$ and by $\mathcal{K}(X)$ the subalgebra of $C^\infty(X)$ consisting of all continuous functions with compact support. For $g \in \mathcal{K}(X)$ and $\beta \in M(X)$ write

$$\langle g, \beta \rangle = \int_X g d\beta.$$

Address delivered before the Chicago Meeting of the Society on April 9, 1965, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors July 5, 1965.

[1] Research supported by the U. S. Army Research Office, Durham, N. C. under contract DA-31-124-ARO-D-288.

1. The existence of a lifting. Let Z be a locally compact space and $\mu \in M_+(Z)$, $\mu \neq 0$. Let ρ be a mapping of $M^\infty(Z, \mu)$ into $M^\infty(Z, \mu)$. Consider the conditions:

(I) $\rho(f) \equiv f$;

(II) $f \equiv g$ implies $\rho(f) = \rho(g)$;

(III) $\rho(1) = 1$;

(IV) $f \geqq 0$ implies $\rho(f) \geqq 0$;

(V) $\rho(af + bg) = a\rho(f) + b\rho(g)$;

(VI) $\rho(fg) = \rho(f)\rho(g)$.

A mapping $\rho: M^\infty(Z, \mu) \to M^\infty(Z, \mu)$ is a *linear lifting* on $M^\infty(Z, \mu)$ if it satisfies the conditions (I)–(V); the mapping ρ is a *lifting* on $M^\infty(Z, \mu)$ if it satisfies (I)–(VI).

Equivalently a linear lifting (respectively a lifting) can be defined as a positive linear mapping (respectively a representation $\tilde{\rho}$) of $L^\infty(Z, \mu)$ into $M^\infty(Z, \mu)$ which satisfies the relations $\tilde{\rho}(\tilde{1}) = 1$ and $\pi \circ \tilde{\rho} =$ the identity (this remark explains the terminology we use).

The problem as to whether or not there exists a lifting on $M^\infty(Z, \mu)$, for $Z = R$ and $\mu =$ the Lebesgue measure on Z, was raised by A. Haar. It was solved by J. von Neumann in a paper published in 1931 ([**50**]); in this paper he established the existence of a lifting in this case.[2] In a subsequent paper published in 1935 ([**53**]) J. von Neumann and M. H. Stone discussed various aspects and generalizations of the problem.

Attempts to solve the problem as to whether or not there exists a lifting on $M^\infty(Z, \mu)$, for arbitrary Z and μ, were unsuccessful until quite recently, although this problem has many relations with various other topics.

Finally, in a paper published in 1958 ([**41**]) D. Maharam established, by a delicate argument, that *a lifting always exists*. D. Maharam proves first the existence of a lifting on $M^\infty(Z, \mu)$ for $Z = \prod_{i \in I} \{a_i, b_i\}$ and $\mu = \bigotimes_{i \in I} \mu_i$, with $\mu_i(\{a_i\}) = \mu_i(\{b_i\}) = 1/2$ for all $i \in I$. Then she reduces the case of arbitrary Z and μ (of σ-finite mass) to this one, via a general isomorphism theorem concerning homogeneous measure algebras ([**6**]; [**40**]).

A different and more direct proof of the existence of a lifting was subsequently given ([**26**]). A variant of this proof can be outlined as follows: We remark first that it is enough to consider the case when Z is compact ([**26**]). We denote by $\mathfrak{A}(Z, \mu)$ the set of all subalgebras $\mathfrak{A} \subset M^\infty(Z, \mu)$ containing 1 and $N^\infty(Z, \mu)$ and satisfying the relation

$$\overline{\pi(\mathfrak{A})} \cap L^\infty(Z, \mu) = \pi(\mathfrak{A});$$

[2] J. von Neumann also considered several other cases that can be reduced to this one.

the closure in this formula is taken in $L^2(Z, \mu)$. Denote by $P_{\mathfrak{A}}$ the restriction to $L^\infty(Z, \mu)$ of the orthogonal projection of $L^2(Z, \mu)$ onto $\pi(\mathfrak{A})$; then $P_{\mathfrak{A}}$ is a Dunford-Schwartz operator[3] (in fact a conditional expectation). Let $\mathfrak{J}$ be the set of all couples $(\mathfrak{A}, \rho^{\mathfrak{A}})$ where $\mathfrak{A} \in \mathfrak{A}(Z, \mu)$ and $\rho^{\mathfrak{A}}: \mathfrak{A} \to \mathfrak{A}$ satisfies (I)–(VI). We order $\mathfrak{J}$ by writing $(\mathfrak{A}, \rho^{\mathfrak{A}}) \leqq (\mathfrak{B}, \rho^{\mathfrak{B}})$ whenever $\mathfrak{A} \subset \mathfrak{B}$ and $\rho^{\mathfrak{B}} | \mathfrak{A} = \rho^{\mathfrak{A}}$ and prove that:

1.1) If $(\mathfrak{A}, \rho^{\mathfrak{A}})$ is maximal in $\mathfrak{J}$ then $\mathfrak{A} = M^\infty(Z, \mu)$;

1.2) $\mathfrak{J}$ is inductive.

The existence of a lifting ρ on $M^\infty(Z, \mu)$ is then established. However the proofs of 1.1) and 1.2) are far from being transparent. For the proof of 1.2) for instance we use, in particular, an analysis of the extremal points of a certain convex set and a pointwise ergodic theorem concerning increasing sequences of projections or a martingale convergence theorem ([27]; [29]).

The following result was given in [26] in connection with the proof of 1.2): (AC) Let Z_1, Z_2 be two compact spaces and let $\mathfrak{C}$ be the convex set of all positive linear mappings T of $C^\infty(Z_1)$ into $C^\infty(Z_2)$ mapping 1 onto 1. Then $T \in \mathfrak{C}$ is extremal if and only if it is multiplicative. This result was subsequently generalized by R. R. Phelps ([55]). Concerning (AC) see also [14] and [59].

Once the existence of a lifting on $M^\infty(Z, \mu)$ is established it is natural to raise the following question: Let $1 \leqq p < \infty$; does there exist a mapping ρ of $\mathfrak{L}^p(Z, \mu)$ into $\mathfrak{L}^p(Z, \mu)$ satisfying the conditions (I), (II), (IV) and (V)?[4] The problem is essentially solved by the following result ([28]): *If $\mu \neq 0$ is diffuse*[5] *then there is no such mapping* ρ. More generally we can prove the following ([30]): *If $\mu \neq 0$ is diffuse then there is no mapping ρ of $\mathfrak{L}^p(Z, \mu)$ into $\mathfrak{L}^p(Z, \mu)$ satisfying* (I), (II), (IV*) *and* (V). Here we denoted by (IV*) the condition: There is $A \subset Z$ with $0 < \mu^*(A) < \infty$ such that $f \to \rho(f)(y)$ is continuous for each $y \in A$.

Let us mention here that it is the positivity of the lifting which makes the proof of its existence hard. It is the same property (or correspondingly the continuity property) which causes the nonexistence of a ⟨⟨linear lifting⟩⟩ on $\mathfrak{L}^p(Z, \mu)$.[6] However it is precisely the *positivity property* of ρ which is important in applications.

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[3] A Dunford-Schwartz operator is a linear mapping T of $L^1(Z, \mu) \cap L^\infty(Z, \mu)$ into itself satisfying the relations $\|T\|_1 \leqq 1$ and $\|T\|_\infty \leqq 1$.

[4] These conditions obviously make sense in this context.

[5] That is $\mu(\{z\}) = 0$ for every $z \in Z$.

[6] The existence of a mapping ρ having the properties (I), (II), (III) and (V) only, can be obtained by a convenient splitting.

2. **The strong lifting.** In various applications it appears necessary to consider liftings having a supplementary property connecting the measure and the topology.

Let Z be a locally compact space and $\mu \in M_+(Z)$, $\mu \neq 0$. A *linear lifting* or a *lifting* ρ on $M^\infty(Z, \mu)$ is *strong* ([31]; [32]) if:

(VII) $\rho(f) = f$ for $f \in C^\infty(Z)$.

It can be shown that a linear lifting ρ on $M^\infty(Z, \mu)$ is strong if $\rho(f) = f$ for every $f \in \mathcal{B}$, where $\mathcal{B} \subset C^\infty(Z)$ is dense in $\mathcal{K}(Z)$ (for the topology induced by the supremum norm). It can be also shown that the existence of a strong linear lifting on $M^\infty(Z, \mu)$ is equivalent with the existence of a strong lifting on $M^\infty(Z, \mu)$ ([31]; [32]).

To shorten the presentation we shall say that the couple (Z, μ) where Z is locally compact and $\mu \in M_+(Z)$, $\mu \neq 0$ has the *strong lifting property* if there exists a strong lifting on $M^\infty(Z, \mu)$. Remark that for (Z, μ) to have the strong lifting property it is of course necessary that $\operatorname{Supp} \mu = Z$. While a lifting always exists (see paragraph 1) the following problem is not yet solved: (A) *Decide whether or not every couple* (Z, μ) (*with* $\mu \neq 0$ *and* $\operatorname{Supp} \mu = Z$) *has the strong lifting property.*

Here are several examples of couples (Z, μ) (we suppose of course $\mu \neq 0$ and $\operatorname{Supp} \mu = Z$) having the strong lifting property:

E_1) Z is metrizable;

E_2) Z is hyperstonean and every rare set is locally μ-negligible;

E_3) $Z = Z_1 \times Z_2$ where Z_1, Z_2 are compact spaces, Z_2 is metrizable and[7] $(Z_1, pr_{Z_1}(\mu))$ has the strong lifting property.

REMARK. The result in E_3) can be improved under supplementary hypotheses. In fact, suppose that for every $pr_{Z_1}(\mu)$-measurable set $A \subset Z_1$ and every open set $U \subset Z_2$ the equation $\mu(A \times U) = 0$ implies that either $pr_{Z_1}(\mu)(A) = 0$ or $pr_{Z_2}(\mu)(U) = 0$. Let ρ be a strong lifting on $M^\infty(Z_1, pr_{Z_1}(\mu))$. Then there is a strong lifting $\tilde{\rho}$ on $M^\infty(Z, \mu)$ satisfying $\tilde{\rho}(f \otimes g) = \rho(f) \otimes g$ for all $f \in M^\infty(Z_1, pr_{Z_1}(\mu))$, $g \in C^\infty(Z_2)$.

E_4) $Z = \prod_{i \in J} Z_i$ where, for each $i \in J$, Z_i is a metrizable compact space and the measure μ satisfies the following condition (for $I \subset J$ we denote by pr_I the canonical projection of Z onto $Z_I = \prod_{i \in I} Z_i$): for every $I \subset J$, $i \in J - I$, A $pr_I(\mu)$-measurable part of Z_I and $B \subset Z_i$ open, the equation

$$\mu(\overset{-1}{pr_I}(A) \cap \overset{-1}{pr_{\{i\}}}(B)) = 0$$

implies that either $\mu(\overset{-1}{pr_I}(A)) = 0$ or $\mu(\overset{-1}{pr_{\{i\}}}(B)) = 0$.

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[7] The notation $pr_{Z_1}(\mu)$ is explained in paragraph 6; pr_{Z_1} is the canonical projection of $Z_1 \times Z_2$ onto Z_1.

E$_5$) $Z=\prod_{i\in J} Z_i$ and $\mu=\bigotimes_{i\in J}\mu_i$ where, for each $i\in J$, Z_i is a metrizable compact space and $\mu_i\in M_+(Z_i)$ and has total mass one.

E$_6$) $Z=[0, 1]^J$ and

$$\mu = \int_{[0,1]} \mu_p d\beta(p).$$

Here $\beta\in M_+([0, 1])$, has total mass one and $\beta(\{0\})=\beta(\{1\})=0$; for each $p\in[0, 1]$, $\mu_p=\prod_{i\in J}\lambda_p^i$ where λ_p^i, $i\in J$, is defined by the equations $\lambda_p^i(\{0\})=p$, $\lambda_p^i(\{1\})=1-p$.

REMARKS. 1) The examples E$_5$) and E$_6$) are particular cases of E$_4$). 2) Some of the above examples are given in [31] and [32].

Let Z be a locally compact polish space,[8] $\mathcal{B}$ the tribe ($=\sigma$-algebra) of all Borel parts of Z and $\mathcal{B}'$ the tribe spanned by the analytic parts of Z. Let $\mu\in M_+(Z)$, $\mu\neq 0$ with Supp $\mu=Z$. We may now formulate the following problem: (B) *Decide whether or not there is a strong lifting on $M^\infty(Z, \mu)$ the range of which consists of functions $\mathcal{B}$-measurable or at least $\mathcal{B}'$-measurable.* A ⟨⟨positive⟩⟩ solution to this problem will be useful in certain applications (with the continuum hypothesis the problem is at least partially solved ([53])). Of course problem (B) can be formulated in a more general setting. Let us remark however that if Z is an arbitrary compact space and $\mu\neq 0$ an arbitrary positive measure on Z with Supp $\mu=Z$ then it is not necessarily true that there is a strong lifting on $M^\infty(Z, \mu)$ the range of which consists of Baire functions.

We shall close this paragraph with one more remark. If (Z, μ) is an arbitrary couple (even having the strong lifting property) then it is not necessarily true that a lifting on $M^\infty(Z, \mu)$ can be modified on a locally μ-negligible set so as to become a strong lifting. In fact (see [32]) there is a couple (Z, μ) with Z compact, having the strong lifting property, and a lifting ρ on $M^\infty(Z, \mu)$ such that

$$Z = \bigcup_{f\in C^\infty(Z)} \{z \mid \rho(f)(z) \neq f(z)\}.$$

3. Liftings commuting with certain groups of mappings. Let Z be a locally compact space and $\mu\in M_+(Z)$, $\mu\neq 0$. Denote by $\mathcal{H}(Z, \mu)$ the group of all bijections $s: Z\to Z$ having the following two properties:

3.1) $s(A)$ and $s^{-1}(A)$ are μ-measurable if A is;

3.2) $s(A)$ and $s^{-1}(A)$ are locally μ-negligible if A is.

We shall denote by e the unit element ($=$the identity mapping). Remark that for each $s\in\mathcal{H}(Z, \mu)$ the mapping $f\to f\circ s$ is an isomorphism

[8] A *polish space* is a metrizable space of countable type on which there exists a metric compatible with its topology, for which the space is complete.

of the algebra $M^\infty(Z, \mu)$ onto itself; clearly $f \equiv g$ implies $f \circ s \equiv g \circ s$.

Let now $G \subset \mathcal{H}(Z, \mu)$. A *linear lifting* or a *lifting* ρ on $M^\infty(Z, \mu)$ commutes with G if:

$$\rho(f \circ s) = \rho(f) \circ s$$

for all $f \in M^\infty(Z, \mu)$ and $s \in G$. Consider now the following problem: (C) *Find conditions on* $\{Z, \mu, G\}$ *which imply the existence of a linear lifting or lifting on* $M^\infty(Z, \mu)$, *commuting with* G.

As we remarked in the previous paragraph the existence of a strong linear lifting is equivalent with the existence of a strong lifting. The situation is no more the same when we consider linear liftings or liftings commuting with various sets $G \subset \mathcal{H}(Z, \mu)$. We have however the following result due to A. Ionescu Tulcea ([**24**]): *Let* $G \subset \mathcal{H}(Z, \mu)$ *be a group having the property*: (p) *For every* $x \in Z$ *the mapping* $s \rightarrow s(x)$ *is injective. Then the existence of a linear lifting on* $M^\infty(Z, \mu)$ *commuting with* G *is equivalent with the existence of a lifting on* $M^\infty(Z, \mu)$ *commuting with* G. Let us remark here that if $Z = R$, $\mu =$ the Lebesgue measure on R and G is the group consisting of the identity mapping and of $x \rightarrow -x$ then there is no lifting on $M^\infty(Z, \mu)$ commuting with G (although there is a strong linear lifting on $M^\infty(Z, \mu)$ commuting with G (see E_7) below)).

We shall describe here several examples of objects $\{Z, \mu, G\}$ for which linear liftings commuting with G exist:

E_7) If (Z, μ) is arbitrary and $G \subset \mathcal{H}(Z, \mu)$ is a countable amenable ([**4**]) group then there is a linear lifting on $M^\infty(Z, \mu)$ commuting with G.

The hypothesis G amenable is satisfied if G is commutative, in particular if G is generated by a single element.

E_8) If $Z = \prod_{i \in J} Z_i$, where $(Z_i)_{i \in J}$ is an arbitrary family of finite groups, μ is a Haar measure on Z and G is the group of left translations of Z then there is a strong lifting on $M^\infty(Z, \mu)$ commuting with G.

The proof of the assertion in E_8) is direct but quite long.

E_9) If Z is a metrizable locally compact group, μ a left Haar measure on Z and G the group of left translations of Z then there is a strong lifting on $M^\infty(Z, \mu)$ commuting with G.

To prove E_9) we consider first the case when Z is a Lie group ([**24**]). Then we use the fact that there is an open subgroup of Z which can be approximated by Lie groups ([**47**]) and the martingale theorem, in a form given by M. Jerison and G. Rabson ([**34**]).

Remarks. 1) The assertion in E_7) is proved in [**24**]. 2) Consider an object $\{Z, \mu, G\}$. If ⟨⟨convenient⟩⟩ derivating systems exist ([**3**];

[13]; [35]; [57]) then the existence of a linear lifting on $M^\infty(Z, \mu)$, commuting with G, can be obtained using a method due to J. Dieudonné ([6]). This method is used in [24] to prove the existence of a (linear) lifting commuting with the group of left translations of Z, when Z is a Lie group. 3) An example due to A. Kolmogorov ([15]) shows that there is a metrizable compact space Z endowed with a positive Radon measure $\mu \neq 0$ and a nonergodic group G of homeomorphisms of Z leaving μ invariant, which has no strictly invariant μ-measurable set A satisfying $0 < \mu(A) < \mu(G)$. This shows that there is no linear lifting ρ on $M^\infty(Z, \mu)$ commuting with G. In fact if such a linear lifting existed and if A were an invariant set then it would follow that there exists a strictly invariant set equivalent to A. 4) If Z is a locally compact group, μ a left Haar measure on Z, G the group of left translations of Z and ρ a linear lifting on $M^\infty(Z, \mu)$ commuting with G then ρ is a strong linear lifting.

4. Lower densities and liftings. Let X be a locally compact space and $\beta \in M_+(X)$. We shall denote by $\mathfrak{I}(\beta)$ the tribe ($=\sigma$-algebra) of all μ-measurable parts of X. For $A \subset X$ we denote by ϕ_A the characteristic function of A. For A and B in $\mathfrak{I}(\beta)$ we write $A \equiv B$ whenever $\phi_A \equiv \phi_B$.

Let Z be a locally compact space and $\mu \in M_+(Z)$, $\mu \neq 0$. Let θ be a mapping of $\mathfrak{I}(\mu)$ into $\mathfrak{I}(\mu)$. Consider the following conditions:

(I′) $\theta(A) \equiv A$;

(II′) $A \equiv B$ implies $\theta(A) = \theta(B)$;

(III′) $\theta(\emptyset) = \emptyset$ and $\theta(Z) = Z$;

(IV′) $\theta(A \cap B) = \theta(A) \cap \theta(B)$;

(V′) $\theta(A \cup B) = \theta(A) \cup \theta(B)$.

A mapping $\theta: \mathfrak{I}(\mu) \to \mathfrak{I}(\mu)$ satisfying the conditions (I′)–(V′) is called a lifting on $\mathfrak{I}(\mu)$. If ρ is a lifting on $M^\infty(Z, \mu)$ and if, for every $A \in \mathfrak{I}(\mu)$, we define

$$4.1) \qquad \theta(A) = \{z \mid \rho(\phi_A)(z) = 1\}$$

then θ is a lifting on $\mathfrak{I}(\mu)$ (remark that $\rho(\phi_A)$ is a characteristic function). Conversely if θ is a lifting on $\mathfrak{I}(\mu)$ then there is a unique lifting ρ on $M^\infty(Z, \mu)$ satisfying 4.1) ([26]). A mapping $\theta: \mathfrak{I}(\mu) \to \mathfrak{I}(\mu)$ satisfying the conditions (I′)–(IV′) is called (for historical reasons) a *lower density*. The classical example is the Lebesgue lower density on R (see also [17]; [18]; [22]; [23]; [41]; [50]; [57]; [60]).

As we have seen above, a lifting on $M^\infty(Z, \mu)$ induces a lower density (in fact a lifting) on $\mathfrak{I}(\mu)$. Conversely once we have defined a lower density on $\mathfrak{I}(\mu)$ it is relatively easy to construct a proof of the existence of a lifting on $M^\infty(Z, \mu)$ ([8], [24]).[9] Let us point out here however that the difficulty in defining in general a lifting or a lower density is essentially the same. And this difficulty increases when we want the lower density to have certain supplementary properties.

We shall now formulate two results concerning liftings in terms of lower densities:

A) Let Z and μ be as above and denote by τ_Z the topology of Z. Let θ be a lower density on $\mathfrak{I}(\mu)$ and let

$$\tau_\theta = \{\theta(A) - N \mid A \in \mathfrak{I}(\mu),\ N \text{ locally } \mu\text{-negligible}\};$$

it can be shown that τ_θ is a topology on Z ([24]). Moreover: *The couple (Z, μ) has the strong lifting property if and only if there is a lower density θ on $\mathfrak{I}(\mu)$ such that $\tau_\theta \supset \tau_Z$.*

B) Consider an object $\{Z, \mu, G\}$ where Z and μ are as above and G is a subgroup of $\mathfrak{H}(Z, \mu)$. A lower density θ on $\mathfrak{I}(\mu)$ commutes with G if $s(\theta(B)) = \theta(s(B))$ for all $s \in G$ and $B \in \mathfrak{I}(\mu)$. We may now state the following: *Suppose that G has the property (p) of paragraph 3. Then the existence of a lifting ρ on $M^\infty(Z, \mu)$ commuting with G is equivalent with the existence of a lower density θ on $\mathfrak{I}(\mu)$ commuting with G.* Combining this result with E_9) we deduce that if Z is a metrizable locally compact group, μ a left Haar measure on Z and G the group of left translations of Z then there is a lower density θ on $\mathfrak{I}(\mu)$ commuting with G.

REMARKS. 1) The idea of defining a topology τ_θ from a lower density was first pointed out to us by J. Oxtoby. It has been exploited by many authors in one context or another ([17]; [18]; [22]; [23]; [43]; [60]). 2) Let ρ be a lifting on $M^\infty(Z, \mu)$ and θ the lifting on $\mathfrak{I}(\mu)$ defined by 4.1); then ρ is a strong lifting if and only if $\theta(U) \supset U$ for every open set $U \subset Z$.

5. **Integral representations of operators.** Let Z be a locally compact space and $\mu \in M_+(Z)$, $\mu \neq 0$. Let F be a Banach space and F' its dual. For every $f: Z \to F'$ and $x \in F$ denote by $\langle x, f\rangle$ the mapping $z \to \langle x, f(z)\rangle$. Denote by $\mathcal{L}^\infty_{F_s'}(Z, \mu)$ the vector space of all $f: Z \to F'$ such that $\langle x, f\rangle \in M^\infty(Z, \mu)$ for each $x \in F$. For f and g in $\mathcal{L}^\infty_{F_s'}(Z, \mu)$ write $f \equiv g$ whenever $\langle x, f\rangle = \langle x, g\rangle$ for each $x \in F$. We define this way an equivalence relation in $\mathcal{L}^\infty_{F_s'}(Z, \mu)$. Denote by $L^\infty_{F_s'}(Z, \mu)$ the corresponding quotient space and by $f \to \dot{f}$ the corresponding canonical mapping. For $f \in \mathcal{L}^\infty_{F_s'}(Z, \mu)$ let

[9] The proofs given by J. von Neumann and by D. Maharam for the existence of a lifting make essential use of lower densities. J. von Neumann also showed how to deduce a lifting on $\mathfrak{I}(\mu)$ from a lower density. The arguments used in [8] and [50] for the same purpose are completely different from his.

$$N_\infty(\dot f) = \inf\{N_\infty(\|g\|) \mid g \in \mathcal{L}^\infty_{F_s'}(Z, \mu), \dot g = \dot f\};$$

here $\|g\|$ is the mapping $z \rightarrow \|g(z)\|$. The mapping $\dot f \rightarrow N_\infty(\dot f)$ is a norm on $L^\infty_{F_s'}(Z, \mu)$ and defines a structure of a Banach space ([28]).

Consider now the following two assertions concerning the couple (Z, μ):

(M) *There exists a linear lifting on* $M^\infty(Z, \mu)$.

(DP) *For every Banach space F and linear continuous mapping $U: L^1(Z, \mu) \rightarrow F'$ there is $g_U \in \mathcal{L}^\infty_{F_s'}(Z, \mu)$ such that $N_\infty(\dot g_U) = \|U\|$ and*

$$\langle x, U\tilde f\rangle = \int_Z f(z)\langle x, g_U\rangle d\mu(z)$$

for all $\tilde f \in L^1(Z, \mu)$ and $x \in F$.

It can be proved that (M) and (DP) are equivalent assertions ([6; 28]). Since the assertion (DP) is precisely the statement of the Dunford-Pettis theorem and since (see paragraph 1) the assertion (M) is always true it follows that the Dunford-Pettis theorem is valid without any countability hypotheses (([26]; [28]); in the papers just quoted the Dunford-Pettis theorem is given in the context of locally convex spaces). We also want to remark here that the mapping $U \rightarrow \dot g_U$ exhibited in the formulation of (DP) is an isomorphism of the Banach space $\mathcal{L}(L^1(Z, \mu), F')$ onto $L^\infty_{F_s'}(Z, \mu)$.

The existence of a lifting can be essentially used to establish integral representations for operators on a space $L^r_E(Z, \mu)$, $1 \leqq r < \infty$, to a space F; here E and F are Banach spaces, or locally convex spaces. In [28] a unified treatment is provided for various classical representation theorems such as the Dunford-Pettis Theorem (already mentioned above), the Dunford-Pettis-Phillips Theorem, the theorem giving the form of various linear compact operators on a space $L^1(Z, \mu)$, the theorem giving the dual of the space $L^r_E(Z, \mu)$, $1 \leqq r < \infty$, and a theorem of N. Dinculeanu and C. Foiaş concerning the representation of ⟨⟨dominated measures⟩⟩ (see also a recent paper by N. Dinculeanu [7]; for additional bibliography see [28]). The existence of a lifting permitted to remove all countability hypotheses in these theorems although at a certain stage it was believed that certain countability hypotheses are necessary is some of these theorems.

6. **Disintegration of measures.** Let X be a locally compact space and $\beta \in M_+(X)$. We shall denote by $\mathcal{C}(X, \beta)$ the set of all families $\mathcal{K} = (K_i)_{i \in J}$ consisting of disjoint compact parts of T and having the properties: 6.1) $T - \bigcup_{i \in J} K_i$ is locally β-negligible; 6.2) for every compact $K \subset L$ the set $\{j \mid K \cap K_j \neq \emptyset\}$ is countable. We have $\mathcal{C}(X, \beta) \neq \emptyset$ for every X and β ([1], Chapter V).

Let now Z and T be two locally compact spaces and let $\nu \in M_+(T)$, $\nu \neq 0$. For every mapping $\lambda: t \to \lambda_t$ of T into $M_+(Z)$ and $g \in \mathcal{K}(Z)$ denote by $\langle g, \lambda \rangle$ the mapping $t \to \langle g, \lambda_t \rangle$. Let now ρ be a lifting on $M^\infty(T, \nu)$. *The mapping* $\lambda: t \to \lambda_t$ *of* T *into* $M_+(Z)$ *is appropriate with respect to* (ν, ρ) *if*:

6.3) $\langle g, \lambda \rangle$ *is essentially* ν*-integrable for each* $g \in \mathcal{K}(Z)$;

6.4) *there is* $(K_i)_{i \in J} \in \mathcal{C}(T, \nu)$ *satisfying the equations*

$$\rho(\phi_{K_i}\langle g, \lambda \rangle) = \rho(\phi_{K_i})\langle g, \lambda \rangle$$

for every $i \in J$ *and every* $g \in \mathcal{K}(Z)$.

Remark that the assertion 6.4) says in fact that λ is ⟨⟨invariant by ρ⟩⟩. For λ appropriate (with respect to (ν, ρ)) we denote by $\int_T \lambda_t d\nu(t)$ the measure $g \to \int_T \langle g, \lambda_t \rangle d\nu(t)$.

Let $\mu \in M_+(Z)$ and $p: Z \to T$. The mapping p is *μ-proper* if it is μ-measurable and if $f \circ p$ is essentially μ-integrable for every $f \in \mathcal{K}(T)$. If p is μ-proper we denote by $p(\mu)$ the measure $f \to \int_T f \circ p d\mu$ on T.

Let now Z and T be two locally compact spaces, $\mu \in M_+(Z)$, $\mu \neq 0$, $p: Z \to T$ a μ-proper mapping and $\nu = p(\mu)$. With these notations we may now introduce the following definition: *The measure* μ *can be disintegrated with respect to* p *if there is a lifting* ρ *on* $M^\infty(T, \nu)$ *and a mapping* $\lambda: t \to \lambda_t$ *of* T *into* $M_+(Z)$ *appropriate with respect to* (ν, ρ) *such that*

$$6.5) \qquad \langle g, \mu \rangle = \int_T \langle g, \lambda_t \rangle \, d\nu(t)$$

for all $g \in \mathcal{K}(Z)$; 6.6) λ_t *is concentrated on* $p^{-1}(\{t\})$ *for* $t \notin N$, *where* N *is a locally* ν*-negligible set (depending on* λ*).*

The main result concerning the disintegration of measures is the following: (I) *If* (T, ν) *has the strong lifting property then* μ *can be disintegrated with respect to* p. (II) *If for every* Z, T, μ, p *(as above) the measure* μ *can be disintegrated with respect to* p, *then every couple* (X, β) *(with* $\beta \neq 0$ *and* Supp $\beta = X$*) has the strong lifting property.*

For (I) see [31] and [32] (where a somewhat more general result is proved). The result in (II) follows from certain arguments in [32]. With the exception of a special result due to G. Mokobodzki, theorem (I) contains essentially all the results in the literature concerning the disintegration of measures on locally compact spaces[10] (N. Bourbaki ([1, Chapter VI]), J. Dieudonné ([5]; [6]), P. R. Halmos ([20])).

The notion of appropriate family as defined here replaces the no-

[10] For Mokobodzki's result see [25]. See also [25] for various remarks concerning liftings, disintegrations of measures and decompositions into ergodic parts.

tion of adequate family as defined by N. Bourbaki, and used by him in the study of the disintegration of measures ([1, Chapter V and Chapter VI]). If (T, ν) has the strong lifting property and ρ is a strong lifting on $M^\infty(Z, \nu)$ then any ν-adequate family of measures is appropriate with respect to (ρ, ν). In most cases when disintegrations have been performed the existence of a strong lifting can be established. On the other hand, in [32], an example is exhibited where the disintegration can be performed in terms of appropriate families but not in terms of adequate families. Before closing we wish to make one more remark. The definition of ⟨⟨disintegration⟩⟩ can be changed by omitting the requirement that λ be appropriate; however if for instance we want formula 6.5) to remain valid for functions $f \in \mathcal{L}^1(Z, \mu)$ (this is true both in the case of appropriate and adequate families) then some supplementary hypothesis on λ is needed ([1, Chapter V]).

7. **Point realizations of endomorphisms of $L^\infty(Z, \mu)$ spaces.** Let $\mathfrak{Z}' = (Z', \mu')$ and $\mathfrak{Z}'' = (Z'', \mu'')$, where Z', Z'' are two compact spaces and $\mu' \in M_+(Z')$, $\mu'' \in M_+(Z'')$. Denote by $\mathfrak{U}(\mathfrak{Z}', \mathfrak{Z}'')$[11] the set of all mappings $u: Z'' \to Z'$ having the following properties: 7.1) $f \circ u$ is μ''-measurable for every $f \in \mathcal{K}(Z')$; 7.2) $u^{-1}(K)$ is μ''-negligible if $K \subset Z'$ is μ'-negligible. Denote by $\mathfrak{R}(\mathfrak{Z}', \mathfrak{Z}'')$ the set of all representations $\phi: L^\infty(Z', \mu') \to L^\infty(Z'', \mu'')$ mapping $\tilde{1}$ onto $\tilde{1}$ and satisfying $\sup \phi(B) = \phi(\sup B)$ for every bounded filtering set $B \subset L^\infty(Z', \mu')$. For $u \in \mathfrak{U}(\mathfrak{Z}', \mathfrak{Z}'')$ define the mapping β_u by the equations: $\beta_u(\tilde{f}) = \tilde{f} \circ u$ for $\tilde{f} \in L^\infty(Z', \mu')$ (by the properties 7.1) and 7.2) the mapping β_u is well defined). It is easy to see that $\beta_u \in \mathfrak{R}(\mathfrak{Z}', \mathfrak{Z}'')$. Moreover, using the existence of a lifting and a theorem concerning appropriate families, we may show that: *The mapping: $u \to \beta_u$ is a surjection of $\mathfrak{U}(\mathfrak{Z}', \mathfrak{Z}'')$ onto $\mathfrak{R}(\mathfrak{Z}', \mathfrak{Z}'')$.*[12]

Therefore every $u \in \mathfrak{R}(\mathfrak{Z}', \mathfrak{Z}'')$ is induced by a point mapping. In a certain sense the result which we just stated is a generalization (in the context of Radon measures) of a theorem of J. von Neumann ([8]; [51]; see also [19] and [54]) to arbitrary compact spaces. In the general case however we may encounter certain phenomena which do not occur in the case of metrizable compact spaces. For instance we may construct an example of two couples $\mathfrak{Z}' = (Z', \mu')$ and $\mathfrak{Z}'' = (Z'', \mu'')$ having the following properties: j) $\mathfrak{Z}'$ and $\mathfrak{Z}''$ have the strong lifting property; jj) there is an isomorphism γ of $L^\infty(Z', \mu')$ onto $L^\infty(Z'', \mu'')$; jjj) there is no measurable mapping $u \in \mathfrak{U}(\mathfrak{Z}', \mathfrak{Z}'')$ satisfying $\gamma = \beta_u$.

[11] The objects $\mathfrak{Z}$ form a category if we define $\mathrm{Hom}(\mathfrak{Z}', \mathfrak{Z}'') = \mathfrak{U}(\mathfrak{Z}'', \mathfrak{Z}')$.

[12] When $\mathfrak{Z}'' = (Z'', \mu'')$ has the strong lifting property and $\mathfrak{Z}' = (Z', \mu')$ is the associated hyperstonean space then, a variant of this result, gives an interesting imbedding of Z'' into Z'.

8. Final remarks. We have chosen in this presentation the setting of Radon measures for several reasons. One of them is the nature of the subjects treated (see paragraphs 2 and 3); another, the applications made here (see paragraphs 6 and 7). However the problem of the existence of a lifting can be considered for ⟨⟨abstract *complete* measure spaces⟩⟩ also.[13] In fact the existence of a lifting can be proved for measure spaces of finite mass ([26]; [41]). From this it is easy to deduce the existence of a lifting in the case of a totally σ-finite measure space and even in the case of more general measure spaces (see for instance the Technical report [30] and the paper by R. Ryan [56]).

Before closing we shall make one more remark. In the definition of a strong lifting (in paragraph 2), the topological structure of Z was of course used; however the structure of a locally compact space was not used. It would be interesting to define and study the notion of ⟨⟨strong lifting⟩⟩ in more general spaces and to analyze the extent to which its applications remain still valid ([16]; [37]; [44]; [45]). Without going into details we shall mention here the following result: Let (Z, T, μ) be a measure space where Z is a polish space, $\mathfrak{T}$ a tribe containing the Borel sets and μ a ⟨⟨positive measure⟩⟩ on $\mathfrak{T}$ of finite mass. Suppose that: 8.1) $\mu(U)\neq 0$ if U is open and nonvoid; 8.2) $A\in\mathfrak{T}$ if and only if there are Borel sets A' and A'' such that $A'\subset A\subset A''$ and $\mu(A''-A')=0$. Then there is a ⟨⟨strong lifting on $M^\infty(Z, \mathfrak{T}, \mu)$⟩⟩.

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