INVOLUTIONS WITH NONZERO ARF INVARIANT

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Browder and Livesay [1] have associated with each differentiable fixed point free involution $T: \Sigma^{2q+1} \to \Sigma^{2q+1}$, where $\Sigma = \Sigma^{2q+1}$ is a homotopy sphere, a "signature" $\sigma(\Sigma, T) \in Z(\equiv 0 \bmod 8)$ if q is odd, or an "Arf invariant" $c(\Sigma, T) \in Z_2$, if q is even. If $q \geqq 3$, then Σ contains a differentiably imbedded $2q$-sphere invariant with respect to T if and only if $\sigma(\Sigma, T) = 0$ or $c(\Sigma, T) = 0$ [1].

S. López de Medrano has constructed for every odd q examples of involutions T with nonzero signature. We prove the following

THEOREM. *For every $k \geqq 1$ there exists a fixed point free differentiable involution T: $\Sigma^{4k+1} \to \Sigma^{4k+1}$ with $c(\Sigma, T) \neq 0$. Here Σ is the "Kervaire homotopy sphere," i.e., the generator of bP_{4k+2}* [3], [4] *if the latter group is $\neq 0$; otherwise it is the standard sphere.*

The author understands that D. Montgomery and C. T. Yang have an example of a differentiable involution on Σ^9 with $c(\Sigma, T) \neq 0$. The fact that there are PL-involutions with $c(\Sigma, T) \neq 0$ on any $(4k+1)$-dimensional sphere follows from the classification of C. T. C. Wall [8].

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1. **Recall of definitions.** Let T be a differentiable (or PL) fixed point free involution on $\Sigma = \Sigma^{4k+1}$. A characteristic manifold N^{4k} is an invariant submanifold such that $\Sigma = A \cup B$, $N = A \cap B$, $B = TA$. There always exists such an N which is $(2k-1)$-connected [1]. Let $G = H_{2k}(N, Z_2) = H_{2k}(N) \otimes Z_2$. For $x, y \in G$, the intersection coefficients $A(x, y) = x \cdot y \in Z_2$, and $B(x, y) = x \cdot Ty$ define nonsingular symmetric bilinear forms on G and

$$(1) \qquad A(x, Ty) = B(x, y).$$

Browder and Livesay [1] define a quadratic form ψ_0: $G \to Z_2$ (if $x \in G$ is represented by an immersed sphere σ in general position with respect to $T\sigma$, then $\psi_0(x)$ is 1 if and only if $\sigma \cap T\sigma$ consists of an odd number of pairs of points). One can also define [7] another qua-

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dratic form $\mu_0\colon G\to Z_2$ by $\mu_0(x\otimes 1)=\frac{1}{2}x\cdot x \bmod 2$, where $x\in H_{2k}(N, Z)$. ($x\cdot x$ is always even). We have [1], [7]

$$\psi_0(x) = \psi_0(Tx), \qquad \mu_0(x) = \mu_0(Tx); \tag{2}$$

$$\psi_0(x) = 0 \tag{3}$$

if x can be represented by an immersed sphere σ disjoint from $T\sigma$;

$$\begin{aligned} \psi_0(x+y) &= \psi_0(x) + \psi_0(y) + B(x,y), \\ \mu_0(x+y) &= \mu_0(x) + \mu_0(y) + A(x,y). \end{aligned} \tag{4}$$

Let $G_T=G/(1+T)G$. By (1) and (2), we can define following [7], a pairing $C\colon G_T\otimes G_T\to Z_2$ and a quadratic form $\mu_T\colon G_T\to Z_2$ by setting

$$C(\bar{x},\bar{y}) = A(x,y) + B(x,y), \qquad \mu_T(\bar{x}) = \mu_0(x) + \psi_0(x), \tag{5}$$

where x, $y\in G$ represent $\bar{x}$, $\bar{y}\in G_T$. Clearly

$$\mu_T(\bar{x}+\bar{y}) = \mu_T(\bar{x}) + \mu_T(\bar{y}) + C(\bar{x},\bar{y}). \tag{6}$$

Let $i_A\colon G=H_{2k}(N, Z_2)\to H_{2k}(A, Z_2)$, $i_B\colon G\to H_{2k}(B, Z_2)$. Then $G=\operatorname{Ker} i_A\oplus\operatorname{Ker} i_B$.

$$T\operatorname{Ker} i_A = \operatorname{Ker} i_B \tag{7}$$

and (see [1])

$$B \text{ is nonsingular on } \operatorname{Ker} i_A, \tag{8}$$

$$A(x,y) = 0 \quad \text{for } x, y\in \operatorname{Ker} i_A. \tag{8'}$$

Moreover,

$$\mu_0(x) = 0 \quad \text{if } x\in\operatorname{Ker} i_A, \tag{9}$$

since $x=y\otimes 1$, $y\in\operatorname{Ker}(H_{2k}(N, Z)\to H_{2k}(A, Z))$ and $y\cdot y=0$. By (7) $\operatorname{Ker} i_A$ maps isomorphically onto G_T and by (8) and (8') C is nonsingular. Let $e_1, \cdots, e_n, f_1, \cdots, f_n\in\operatorname{Ker} i_A$ be a symplectic basis for $B|\operatorname{Ker} i_A$. Then $\bar{e}_1, \cdots, \bar{e}_n, \bar{f}_1, \cdots, \bar{f}_n\in G_T$ form a symplectic basis for C and (5) and (9) imply

$$c(\Sigma, T) = \Sigma_i\,\psi_0(e_i)\psi_0(f_i) = c_T = \Sigma_i\,\mu_T(\bar{e}_i)\mu_T(\bar{f}_i). \tag{10}$$

The advantage of the identity $c(\Sigma, T)=c_T$ is that c_T is independent of the choice of the symplectic basis $\bar{e}_i$, $\bar{f}_i\in G_T$ [7], whereas $c(\Sigma, T)$ is independent of the choice of the characteristic submanifold N [1].

2. **The involution** T. We shall follow here a construction described in [6]. Let $S^{2k+1}\subset R^{2k+1}$ be the unit sphere $\|x\|=1$, where $x=(x_0, x, \cdots, x_{2k+1})$ and let $S^{2k}\subset S^{2k+1}$ be the "meridian" $x_{2k+1}=0$. Define the rotation $\rho\colon S^{2k+1}\to S^{2k+1}$

$$\rho(x_0, x_1, \cdots, x_{2k+1}) = (x_0, -x_1, \cdots, -x_{2k+1})$$

with fixed points $P=(1, 0, \cdots, 0)$ and $Q=(-1, 0, \cdots, 0)$; $\rho(S^{2k}) = S^{2k}$. Let λ: $[-1, 1]\to[0, 1]=I$ be a C^∞ function such that $\lambda(s)=0$ for $s\geqq\epsilon$ and $\lambda(s)=1$ for $s\leqq-\epsilon$. Define

$$X \subset S^{2k+1} \times S^{2k+1} \times I$$

to be the set of points $(x, y, 0)$ such that $d(x, y)\leqq\epsilon$ (where d is the natural Riemannian metric of S^{2k+1}) and

$$X' \subset S^{2k+1} \times S^{2k+1} \times I$$

the set of points $(x, y, \lambda(x_0))$ such that $d(x, \rho(y))\leqq\epsilon$. Both X and X' are diffeomorphic to the total space of the disk tangent bundle of S^{2k+1} and $X\cap X'$ is a neighborhood of $(P, P, 0)$. Let $Y=X\cup X'$. Define the involution T: $Y\to Y$ by

$$T(x, y, t) = (\rho(x), \rho(y), t).$$

Then the only fixed points of T are $(P, P, 0)$, $(Q, Q, 0)$ and $(Q, Q, 1)$. After straightening the corners (this can be done in a way compatible with T), Y becomes a differentiable manifold with boundary $\Sigma=\Sigma^{4k+1}$ where Σ is the Kervaire homotopy sphere and $T|\Sigma$ has no fixed points. However, since the presence of corners does not affect the value of $c(\Sigma, T)$ we shall continue to use the initial explicit description of Y and of $\Sigma=\partial Y$.

Let $V\subset Y$ be the set of $(x, y, t)\in Y$ such that $x\in S^{2k}$. Then $TV=V$ and V is a $(4k+1)$-manifold with $\partial V=N=\Sigma\cap V$. Let also $R = \{(x, y, t)\mid x_{2k+1}\geqq 0\}$. Then $Y=R\cup TR$, $V=R\cap TR$. Finally, if $A=\Sigma\cap R$, $B=\Sigma\cap TR$, then $\Sigma=A\cup B$, $N=A\cap B$, so that N is a characteristic manifold for (Σ, T).

Let $W\subset V$ be the set of (x, y, t), $x, y\in S^{2k}$. Then W consists of two copies of the tangent disk bundle of S^{2k}, "plumbed" together in a neighborhood of $(P, P, 0)$. The only nonvanishing reduced homology group of W is $H_{2k}(W, Z)=Z+Z$; the two generators e, f are represented by the imbeddings $S^{2k}\to W$ by $(x, x, 0)$ and $(x, \rho(x), \lambda(x_0))$, $x\in S^{2k}$. The intersection coefficients are

$$e\cdot e = f\cdot f = 2, \qquad e\cdot f = 1. \tag{11}$$

For $y\in S^{2k+1}$ such that $d(S^{2k}, y)\leqq\epsilon$, let $p(y)\in S^{2k}$ be the point in which the great circle through y orthogonal to S^{2k} meets S^{2k}. Let $W'\subset\partial V=N$ be the set of $(x, y, t)\in N$ with $y_{2k+1}\geqq 0$ and $W''\subset N$ the set of $(x, y, t)\in N$ with $y_{2k+1}\leqq 0$. Then the correspondence $(x, y, t)\to(x, p(y), t)$ is a homeomorphism of W' and of W'' onto W. Moreover, $N=W'\cup W''$, $\partial W=W'\cap W''$ and $TW'=W''$. In other words N is the double of W and T maps one copy of W onto the other.

3. **Computation of** $c(\Sigma, T)$. Lefschetz duality and the exactness of the homology sequence of the pair $(W, \partial W)$, together with (11), imply that $H_i(\partial W, Z)=0$, $i\neq 0$, $2k-1$, $4k-1$ and that $H_{2k-1}(\partial W)=Z_3$. Therefore the inclusion

$$(12) \qquad j: H_{2k}(W', Z_2) \oplus H_{2k}(W'', Z_2) \xrightarrow{\approx} H_{2k}(N, Z_2) = G$$

is an isomorphism (which preserves intersections). By (11) and (12), G has a basis e', f', Te', Tf' represented by e, f in W' and their images by T in W'' and

$$(13) \qquad A(e',f') = 1, \qquad A(e',e') = A(f',f') = 0$$

and clearly

$$(14) \qquad B(e',f') = B(e',e') = B(f',f') = 0.$$

Moreover, since $\frac{1}{2}e\cdot e=\frac{1}{2}f\cdot f=1$,

$$(15) \qquad \mu_0(e') = \mu_0(f') = 1,$$

whereas (3) implies that

$$(16) \qquad \psi_0(e') = \psi_0(f') = 0.$$

As a consequence of (13) and (14) the images $\bar{e}, \bar{f}\in G_T$ of e', f' form a symplectic basis of G_T with respect to C and (15) and (16) imply that $\mu_T(\bar{e})=\mu_T(\bar{f})=1$ so that by (10)

$$c(\Sigma, T) = c_T = 1.$$

REMARK 1. Since $bP_6=bP_{14}=0$ [4], [6] and also $bP_{30}=0$ (by the recent work of Browder), there are fixed point free involutions with Arf invariant 1 on S^5, S^{13} and S^{29}.

REMARK 2. It would be interesting to know what the relation is between the $T: S^5\to S^5$ constructed in this paper and the nonstandard involution on S^5 described in [2]. However, it follows from [8] that our involution is equivalent to a generator of the group Z_4 of fixed point free involutions on S^5.

After this paper has been completed, the author learned that the existence of differentiable involutions on a homotopy sphere Σ^{4k+1} for all k with $c(\Sigma, T)\neq 0$ has been proven by an entirely different method by W. Browder (not yet published). Such an example on Σ^9 was also obtained by D. Sullivan.

Added in proof. A PL-classification of fixed point free involutions analogous to that of [8], which also implies the existence of PL-involutions with $c(\Sigma, T)=0$ has been obtained independently by S. López de Medrano (to appear in the Proceedings of the Tulane Conference on Transformation Groups, 1967).

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