

SCHAUDER DECOMPOSITIONS IN BANACH SPACES¹

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A sequence (M_i) of closed subspaces of a Banach space E is called a Schauder decomposition of $[M_i]$, the smallest subspace containing $\cup M_i$, if every element u of $[M_i]$ has a unique, norm convergent expansion $u = \sum u_i$, where $u_i \in M_i$ for $i=1, 2, \dots$. It is well known (see, e.g. [2]) that any sequence $(u_i) \subset E$ with $0 \neq u_i \in M_i$ for $i=1, 2, \dots$ is basic (i.e., a basis for its closed linear span). The converse of this statement is not true, but we do derive the following theorem, and mention several corollaries.

THEOREM. *Let (M_i) be a sequence of closed subspace of the Banach space E such that each sequence $(u_i) \subset E$ with $0 \neq u_i \in M_i$ is basic. Then there exists an integer N such that $(M_i | i \geq N)$ is a Schauder decomposition of $[M_i | i \geq N]$.*

To simplify the proof of the theorem, we use the following characterization of Schauder decompositions due to Grinblyum [3]. A sequence (M_i) of closed subspaces of E is a Schauder decomposition of $[M_i]$ if and only if there exists a constant K such that for all integers n, m and all sequence (u_i) with $u_i \in M_i$, $\|\sum_{i=1}^n u_i\| \leq K \|\sum_{i=1}^{n+m} u_i\|$. We note that this norm condition may be replaced by $\|\sum_{i=1}^n a_i u_i\| \leq K \|\sum_{i=1}^{n+m} a_i u_i\|$ where the scalars (a_i) are also arbitrary. Since a sequence $U = (u_i)$ is basic if and only if there exists $K = K(U)$ such that this last inequality holds for all (a_i) , m and n , we see that each (u_i) with $u_i \in M_i$ is basic if (M_i) is a Schauder decomposition.

Let $U = (u_i)$ be a sequence with $0 \neq u_i \in M_i$, and set $U_n = (u_i | i \geq n)$. Let $K(U_n)$ be the smallest constant such that $\|\sum_{i=n}^p a_i u_i\| \leq K \|\sum_{i=n}^{p+q} a_i u_i\|$ holds for all $K \geq K(U_n)$, all (a_i) and integers p, q .

LEMMA. *Let (M_i) be a sequence of closed subspaces of E such that each $U = (u_i)$ with $0 \neq u_i \in M_i$ is basic. Then there exists an integer N and a constant $K \geq 1$ such that every sequence U as above has $K(U_N) \leq K$.*

PROOF. If K and N do not exist, then for each integer n and each $M \geq 1$, there exists a U with $K(U_n) > M$ (noting $K(U_{n+1}) \leq K(U_n)$). Choose $U^{(1)}$ so that $K(U^{(1)}) > 2$. Then there exist integers $q_1 > p_1$ such that $\|\sum_{j=1}^{q_1} a_j u_j^{(1)}\| > 2 \|\sum_{j=1}^{p_1} a_j u_j^{(1)}\|$ for some sequence (a_i) . Similarly, there exist $U^{(2)}$ and $q_2 > p_2$ such that

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$$\left\| \sum_{j=q_1+1}^{p_1} a_j u_j^{(2)} \right\| > 4 \left\| \sum_{j=q_1+1}^{q_2} a_j u_j^{(2)} \right\|,$$

and in general we get $U^{(j)}$ and integers p_j, q_j such that $q_{j-1} < p_j < q_j$ and

$$\left\| \sum_{i=q_{j-1}+1}^{p_j} a_i u_i^{(j)} \right\| > 2^j \left\| \sum_{i=q_{j-1}+1}^{q_j} a_i u_i^{(j)} \right\|.$$

With these bounds, the sequence U defined by $u_i = u_i^{(j)}$ if $q_{j-1} < i \leq q_j$ is not basic, which is a contradiction proving the lemma.

The theorem follows immediately from the lemma and the Grinblyum criterion.

To see that N is in general greater than 1, let E be separable, (x_i) a basic sequence in E such that $\text{codim } [x_i] = \infty$ and E_1 a closed subspace of E which is a quasicomplement but not a complement of $[x_i]$ in E . (For a construction of such an E_1 see Gurarii and Kadec [4].) If we set $M_1 = E_1, M_2 = [x_1], M_3 = [x_2], \text{ etc.}$, each sequence with just one element in each M_i is basic, but (M_i) is not a Schauder decomposition of E since $M_1 + [x_i] \subsetneq E$. In order to have $N = 1$, then, we must keep $[M_i | i < N]$ from being a quasicomplement of $[M_i | i \geq N]$ for each N . In fact, the addition of this hypothesis is also sufficient, for then we see that $[M_i] = M_1 \oplus M_2 \oplus \dots \oplus M_{N-1} \oplus [M_i | i \geq N]$, and so (M_i) is a Schauder decomposition of $[M_i]$. These corollaries are now immediate. In each, we let U be an arbitrary sequence (u_i) with $0 \neq u_i \in M_i$, and call U a proper sequence.

COROLLARY. *A sequence (M_i) of closed subspaces of E is a Schauder decomposition if and only if (a) $[M_i] = [M_i | i < n] \oplus [M_i | i \leq n]$ and (b) each proper sequence U is basic.*

COROLLARY. *The previous corollary holds with (a) replaced by (a') $[M_i] = M_k \oplus [M_i | i \neq k]$ for each k .*

COROLLARY. *Let (M_i) be a sequence of finite-dimensional subspaces of E . Then (M_i) is a Schauder decomposition if and only if each proper sequence is basic.*

It is easy to see that an N dimensional Banach space F has a basis $(f_i)_{i=1}^N$ such that

$$\left\| \sum_{i=1}^p a_i f_i \right\| \leq N \left\| \sum_{i=1}^N a_i f_i \right\|$$

always holds (using, for example, the result of Taylor [5]). The author does not know what the best bound that can replace N in general will be, but it must be greater than 1 (see, e.g. [1]). However, using the last corollary, and the N -bound above, we obtain:

PROPOSITION. Let $\dim M_i \leq N$, $M_i \in E$ for $i=1, 2, \dots$, and $[M_i] = E$. Set $N_j = \dim M_1 + \dim M_2 + \dots + \dim M_j$. The following are equivalent

- (a) (M_i) is a Schauder decomposition of E ,
- (b) E has a basis (x_i) with $M_j = [x_i | N_{j-1} < i \leq N_j]$,
- (c) each proper sequence is basic.

The proof of the proposition is a routine exercise.

PROBLEM. Does the previous result hold with the weaker assumption $\dim M_i < \infty$?

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