EXTENSION OF A THEOREM OF CARLESON

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One of the main ingredients in Carleson's solution to the corona problem [2] is the theorem characterizing the measures μ on the open unit disk with the property that $f \in H^p$ implies

$$\int_{|z|<1} |f(z)|^p d\mu(z) < \infty, \qquad 0 < p < \infty.$$

Carleson's proof of this theorem involves a difficult covering argument and the consideration of a certain quadratic form (see also [1]). L. Hörmander later found a proof which appeals to the Marcinkiewicz interpolation theorem and avoids any discussion of quadratic forms. The main difficulty in this approach is to show that a certain sublinear operator is of weak type (1, 1). Here a covering argument reappears which is similar to Carleson's but apparently easier (see [4]).

We wish to point out that Hörmander's argument, with appropriate modifications, actually proves the theorem in the following extended form.

THEOREM. *Let μ be a finite measure on $|z| < 1$, and suppose $0 < p \leqq q < \infty$. Then in order that there exist a constant C such that*

$$\left\{\int_{|z|<1} |f(z)|^q d\mu(z)\right\}^{1/q} \leqq C\|f\|_p \tag{1}$$

for all $f \in H^p$, it is necessary and sufficient that there be a constant A such that

$$\mu(S) \leqq A h^{q/p} \tag{2}$$

for every set S of the form

$$S = \{re^{i\theta}: 1 - h \leqq r < 1,\ \theta_0 \leqq \theta \leqq \theta_0 + h\}. \tag{3}$$

OUTLINE OF PROOF. A standard argument (factoring out Blaschke products) shows it is enough to consider the case $p = 2$. The necessity of (2) is then proved by choosing $f(z) = (1 - \alpha z)^{-1}$, where $|\alpha| < 1$.

Conversely, let $p = 2$ and suppose (2) holds. Since each $f \in H^2$ is the Poisson integral of its boundary function, it will be sufficient to prove that

$$\left\{\int_{|z|<1} [u(z)]^q d\mu(z)\right\}^{1/q} \leq C\|\varphi\|_2 \tag{4}$$

if $u(z)$ is the Poisson integral of a nonnegative function $\varphi \in L^2$.

With each point $z = re^{i\theta}$ in $0 < |z| < 1$ we associate the boundary arc

$$I_z = \left\{e^{it}: \theta - \tfrac{1}{2}(1 - r) \leq t \leq \theta + \tfrac{1}{2}(1 - r)\right\}.$$

Taking $0 \leq \theta < 2\pi$, we can identify I_z with a segment on the real line. Given an integrable function $\varphi(t) \geq 0$, periodic with period 2π, define

$$\tilde{\varphi}(z) = \sup \frac{1}{|I|} \int_I \varphi(t) dt,$$

where the supremum is taken over all intervals $I \supset I_z$ of length $|I| < 1$. Then $\tilde{\varphi}(z)$ is continuous in $0 < |z| < 1$. It is not difficult to show that

$$u(z) \leq 16\pi^2\{\tilde{\varphi}(z) + \|\varphi\|_1\}, \qquad |z| < 1,$$

where u is the Poisson integral of φ. Thus it will suffice to prove (4) with $\tilde{\varphi}$ replacing u.

In other words, we must show that the sublinear operator $T: \varphi \rightarrow \tilde{\varphi}$ is of type $(2, q)$. Since T is trivially of type (∞, ∞), this will follow from the Marcinkiewicz interpolation theorem if it can be shown that T is of weak type $(1, q)$ for $1 \leq q < \infty$:

$$\mu(E_s) \leq Cs^{-q}\|\varphi\|_1^q, \tag{5}$$

where

$$E_s = \{z: \tilde{\varphi}(z) > s\}, \qquad s > 0.$$

It is only in proving (5) that any use is made of the assumption that

$$\mu(S) \leq Ah^q \tag{6}$$

for all S of the form (3). (For convenience, $q/2$ has been replaced by q.)

Essentially following Hörmander [4], we define for each $\epsilon > 0$ the sets

$$A_s^\epsilon = \left\{z: \int_{I_z} |\varphi(t)| dt > s(\epsilon + |I_z|)\right\},$$

and

$$B_s^\epsilon = \{z: I_z \subset I_w \text{ for some } w \in A_s^\epsilon\}.$$

Note that

$$\mu(E_s) = \lim_{\epsilon\to 0} \mu(B_s^\epsilon). \tag{7}$$

If $z_n \in A_s^\epsilon$ and the arcs I_{z_n} are disjoint, then

$$s\sum_n (\epsilon + |I_{z_n}|) < \sum_n \int_{I_{z_n}} |\varphi(t)|\,dt \leqq 2\pi\|\varphi\|_1. \tag{8}$$

In particular, there can be at most a finite number of points z_n in A_s^ϵ whose associated arcs I_{z_n} are disjoint. The following lemma, whose proof we omit, is now needed (compare [4, Lemma 2.2]).

COVERING LEMMA. *Let A be a nonempty set in $|z|<1$ which contains no infinite sequence of points whose associated arcs I_{z_n} are disjoint. Then there exists a finite number of points $z_1, \cdots, z_m$ in A such that the arcs I_{z_n} are disjoint and*

$$A \subset \bigcup_{n=1}^{m} \{z\colon I_z \subset J_{z_n}\},$$

where J_z is the arc of length $5|I_z|$ whose center coincides with that of I_z.

If E_s is nonempty, the lemma gives (for some $\epsilon>0$)

$$A_s^\epsilon \subset \bigcup_{n=1}^{m} \{z\colon I_z \subset J_{z_n}\},$$

where $z_n \in A_s^\epsilon$ and the arcs I_{z_n} are disjoint. It follows that

$$B_s^\epsilon \subset \bigcup_{n=1}^{m} \{z\colon I_z \subset J_{z_n}\}.$$

Thus the hypothesis (6) gives

$$\mu(B_s^\epsilon) \leqq C\sum_{n=1}^{m} |I_{z_n}|^q. \tag{9}$$

But by (8) we have (since $q \geqq 1$)

$$\left\{\sum_{n=1}^{m} |I_{z_n}|^q\right\}^{1/q} \leqq \sum_{n=1}^{m} |I_{z_n}| < 2\pi s^{-1}\|\varphi\|_1.$$

This together with (7) and (9) proves (5), and (1) follows.

Two applications are worth noting:

1. If $0<p<q<\infty$, then $f\in H^p$ implies

$$\int_0^1 (1-r)^{q/p-2} M_q^q(r,f)\,dr<\infty,$$

where

$$M_q^q(r,f)=\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q d\theta.$$

This useful result is due to Hardy and Littlewood [**3**].

2. If $0<p\leqq q<\infty$, and $f\in H^p$, then

$$\left\{\int_{-1}^1 (1-r)^{q/p-1}|f(r)|^q dr\right\}^{1/q}\leqq C\|f\|_p.$$

This is a generalization of the Fejér-Riesz theorem, aside from the value of the constant C.

References

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