A PROOF OF THE SLICING THEOREM FOR 2-SPHERES

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1. **Introduction.** The purpose of this note is to answer in the affirmative the following question raised by Bing [3]: *Is a 2-sphere S in Euclidean 3-space E^3 tame if each horizontal cross-section of S is either a point or a simple closed curve?* (I have been informed by R. H. Bing that W. T. Eaton has independently proved a similar theorem.)

If S is polyhedral that S is flat is well known. A proof is sketched in [1] and more detailed proofs are given in [5] and [6]. However, the more general assertion posed by the above question was placed in doubt by an example due to Bing. In [3, p. 362] a nonpolyhedral, tame 2-sphere is constructed which satisfies the above condition but cannot be taken onto a round sphere by a level preserving homeomorphism of E^3.

The proof given here is not elementary in that it relies on two characterizations, due to Bing, [2] and [4], of tame 2-spheres in E^3. More will be said about these results when they are used.

I would like to thank R. H. Bing and I. Ferris for helpful comments.

We assume the usual metric on E^3 throughout this paper and in order to simplify the notation we let

$$E_t = \{(x, y, z) \in E^3 \mid z = t\}.$$

2. THEOREM. *Let S^2 be a 2-sphere in E^3 and $a<b$ real numbers such that for $a<t<b$, $S^2 \cap E_t = J_t$ is a simple closed curve and $S^2 \cap E_t = p_t$ is a point for t equal to a or b. Then S^2 is tame.*

PROOF. We first prove that S^2 is locally tame modulo $\{p_a, p_b\}$ by showing that the complementary domains of S^2 are locally simply connected at each point $p \in S^2 - \{p_a, p_b\}$, [2]. The two extreme points will be dealt with in the last step of the proof.

Given $p \in S^2 - \{p_a, p_b\}$ let U be a round open ball containing p and small enough so that no J_t lies entirely in U. Further assume that U misses E_a and E_b. Choose V to be a round open ball containing p such that $V \subset U$ and $V \cap S^2$ lies in a disk in $U \cap S^2$. Let D be a 2-cell. We want to show that any map

$$f\colon \operatorname{Bd} D \to V - S^2$$

admits an extension

$$g: D \to U - S^2.$$

We can assume that $f(\text{Bd } D)$ is a polygonal simple closed curve such that no edge lies on a horizontal plane and that $f(\text{Bd } D) \subset \text{Int } S^2 \cap V$. Since V is simply connected there exists a piecewise linear map

$$\hat{h}: D \to V$$

such that

$$\hat{h}|_{\text{Bd } D} = f \quad \text{on Bd } D.$$

Let C be the component of $D - \hat{h}^{-1}(\hat{h}(D) \cap S^2)$ containing Bd D. Since $\hat{h}(D) \cap S^2$ lies in a disk in $U \cap S^2$ there exists a map

$$h: D \to U \cap \text{Cl}(\text{Int } S^2)$$

such that

$$h|_C = \hat{h} \quad \text{on } C.$$

Also

$$h|_{\text{Bd } D} = f \quad \text{on Bd } D.$$

Since h is piecewise linear on a closed neighborhood in D of Bd D, for a fixed value t, only finitely many components of the set $h^{-1}(h(D) \cap E_t)$ will have the property that each contains more than one point of Bd D. We call the components with this property $A_1^t, \cdots, A_{k(t)}^t$.

The remainder of the proof of this part of the theorem is divided into four parts.

(A) In this part each set $h(A_i^t)$ is replaced by a certain singular finite graph. We begin by pushing J_t slightly into Int J_t on E_t. Since $S^2 \cap E_t = J_t$, a simple closed curve, there exists a homeomorphism

$$k: E_t \to E_t$$

such that

$$k|_{h(\text{Bd } D) \cap E_t} = 1$$

and

$$k(h(D) \cap E_t) \subset U \cap \text{Int } S^2 \cap E_t.$$

Let $O_j \subset U \cap \text{Int } S^2 \cap E_t$ be a connected open set about $kh(A_j^t)$ $(j = 1, \cdots, k(t))$. Let p be some point in O_j. We join each point $q \in h(\text{Bd } D) \cap O_j$ to p by an arc $[pq] \subset O_j$ and take K_j^t to be the union of the finite number of arcs in O_j obtained in this way. The arcs $[pq]$ may intersect each other badly.

(B) We now select a finite number of planes $E_{t_1}, \cdots, E_{t_n}$ so as to satisfy a certain homotopy condition. Let $K^t = K_1^t \cup \cdots \cup K_{k(t)}^t$. For each t $(a<t<b)$ let ϵ_t be a positive number such that $\epsilon_t < \rho(K^t, S^2)$ and $\epsilon_t < \rho(f(\text{Bd } D), S^2)$. Let $V_{t'}(K^t)$ be the vertical translate of K^t to the horizontal plane $E_{t'}$. Then $V_{t'}(K^t) \cap S^2 = \emptyset$ for $|t'-t| < \epsilon_t$.

Let $e<f$ be real numbers such that $h(\text{Bd } D) \subset \bigcup_{e \le t \le f} E_t$. We cover the interval $[e, f]$ by open intervals $W_t = (t-\epsilon_t, t+\epsilon_t)$, $e \leqq t \leqq f$. Let $W_{t_1}, \cdots, W_{t_n}$ be a finite subcollection of the W_t covering $[e, f]$ indexed so that $t_1 < t_2 < \cdots < t_n$. Let t_i be a number in $W_{t_i} \cap W_{t_{i+1}}$. Then each map of a simple closed curve into

$$K^{t_i} \cup K^{t_{i+1}} \cup \Big(h(\text{Bd } D) \cap \bigcup_{t_i \le t \le t_{i+1}} E_t\Big)$$

is homotopic to a map of the simple closed curve into $E_{t_i} \cap \text{Int } S^2 \cap U$.

We now prove the following: given any map

$$d\colon \text{Bd } D \to E_t^2 \cap \text{Int } S^2 \cap U$$

there exists a map

$$\hat{d}\colon D \to E_t^2 \cap \text{Int } S^2 \cap U$$

such that $\hat{d}$ is an extension of d.

Recall that $E_t \cap S^2 = J_t$, a simple closed curve. Let J_t' be a simple closed curve in $\text{Int } J_t$ on E_t such that

(i) $d(\text{Bd } D) \subset \text{Int } J_t'$

(ii) $J_t' \cap U$ consists of a finite number of open arcs $B_1, \cdots, B_i$. Since $U \cap E_t$ is a disk, the map d admits an extension $d'\colon D \to U \cap E_t$. Let C be the component of $D-(d')^{-1}(d'(D) \cap J_t')$ containing Bd D. Let F be a component of the boundary of C relative to D. The set $d'(F)$ must lie in one of the arcs B_j. Therefore $d'|_C$ can be extended to map F and the components in Int D of $D-F$ into B_j. In this way, we extend $d'|_C$ to all of D and obtain the required extension

$$\hat{d}\colon D \to U \cap \text{Cl}(\text{Int } J_t') \subset U \cap E_t \cap \text{Int } S^2.$$

Thus we have shown the following: each map of a simple closed curve into

$$K^{t_i} \cup K^{t_{i+1}} \cup \Big(h(\text{Bd } D) \cap \bigcup_{t_i \le t \le t_{i+1}} E_t\Big)$$

is homotopic in $U \cap \text{Int } S^2$ to a constant map.

(C) Now consider the disjoint connected sets A_j^i in D each of which contains at least two points of Bd D. Let O_j^t, $(i=t_1, \cdots, t_n;$ $j=1, \cdots, k(t_n))$ be a collection of disjoint open subsets of D such that $A_j^t \subset O_j^t$.

In each O_j^t choose a point $p \in O_j^t \cap \text{Int } D$. For each $q \in A_j^t \cap \text{Bd } D$ we take an arc $[pq]$ in O_j^t such that $[pq] \cap [pq'] = p$ for $q' \neq q$ and $q, q' \in A_j^t \cap \text{Bd } D$. Assume also that $[pq] \cap \text{Bd } D = q$. Let L_j^t be the finite graph (a "spoke" with p as center) in O_j^t formed by the union of all such arcs in O_j^t.

(D) Let $L^t = L_1^t \cup \cdots \cup L_{k(t)}^t$ for $t = t_1, \cdots, t_n$. Consider a component M of $D - \bigcup_{i=1}^n L^{t_i}$. The closure of M is a topological disk. We claim that the boundary of M in D must lie in $L^{t_i} \cup L^{t_{i+1}}$ for some $1 \leqq i \leqq n-1$. Consider t_a, t_b, t_c such that $1 \leqq a < b < c \leqq n$. Then $h^{-1}(h(D) \cap E_{t_b})$ separates any $A_i^{t_a}$ $(i=1, \cdots, k(t_a))$ from any $A_j^{t_c}$, $(j=1, \cdots, k(t_c))$.

Now fix i and j. Since $A_i^{t_a}$ and $A_j^{t_c}$ are connected sets and contain points of Bd D some component of $h^{-1}(h(D) \cap E_{t_b})$ containing at least two points of Bd D must separate $A_i^{t_a}$ from $A_j^{t_c}$. But such a component is one of the sets $A_m^{t_b}$, $(1 \leqq m \leqq k(t_b))$. Therefore $L_m^{t_b}$ separates $L_i^{t_a}$ from $L_j^{t_c}$ in D and our claim is verified.

We now extend f to a map f_1 such that

$$f_1\colon L_i^t \to K_i^t; \qquad (t = t_1, \cdots, t_n; i = 1, \cdots, k(t_n)).$$

Again let M be a component of $D - \bigcup_{i=1} L^{t_i}$. Then

$$f_1(\text{Bd } M) \subset K^{t_i} \cup K^{t_{i+1}} \cup (f(\text{Bd } D) \cap \bigcup_{t_i \leq t \leq t_{i+1}} E_t)$$

for some i. Thus, by part (B), f_1 can be extended to map M into $U \cap \text{Int } S^2$. By extending f_1 over all such components M we obtain a map

$$g\colon D \to U \cap \text{Int } S^2$$

such that

$$g|_{\text{Bd } D} = f \quad \text{on Bd } D.$$

This completes the proof that $S^2 - \{p_a, p_b\}$ is locally tame.

(E). We finish the proof of the theorem by showing that S^2 can be homeomorphically approximated by 2-spheres from either complementary domain and is therefore tame, [4]. Let $\epsilon > 0$ be given. We first consider the problem of approximating from the exterior of S^2. Choose U to be a round open ball about p_b of diameter less than $\epsilon/2$. Choose t' such that $J_t \subset U$ for $t' \leqq t < b$. On $E_{t'}$, the simple closed curves $J_{t'}$ and Bd $U \cap E_{t'}$ bound an annulus A. We now discard that part of S^2 lying above $E_{t'}$ and add to remainder of S^2 the annulus A plus the portion of Bd U lying above $E_{t'}$. By performing a similar modification about p_a we obtain a 2-sphere $\hat{S}^2$ such that

$$\{p_a, p_b\} \subset \text{Int } \hat{S}^2 \text{ and } S^2 \subset \text{Cl}(\text{Int } \hat{S}^2).$$

Further $\hat{S}^2$ is tame as it is locally tame modulo two tame simple closed curves. Thus by pushing $\hat{S}^2$ slightly into its exterior we obtain the desired ϵ-approximation.

The interior approximation is even easier. We choose curves J_{t_1} and J_{t_2} sufficiently close to p_a and p_b respectively, throw away the parts of S^2 lying below E_{t_1} and above E_{t_2}, add Int J_{t_1} and Int J_{t_2} and then push our modified tame sphere slightly into its interior.

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