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UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CHAPEL HILL, NORTH CAROLINA 27514 AND
PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

MAXIMAL FUNCTIONS FOR A CLASS OF LOCALLY COMPACT NONCOMPACT GROUPS

BY KEITH PHILLIPS

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In this note, we briefly describe some maximal theorem results to be proved in detail in an appendix (§4) to the paper [PT]. In [PT], maximal averages taken over sets of unbounded measure for functions of several variables over a local field are used to study singular integrals. The results on maximal functions can, however, be obtained for a large class of topological groups, and it is these results which we will describe. The results generalize theorems on maximal functions appearing in [EH], where the sets over which averages are taken have bounded measures. Let Z denote the integers. Our hypothesis is that G is a locally compact group (written multiplicatively) with left Haar measure λ and that $\{U_n : n \in Z\}$ is a neighborhood base at the identity e consisting of relatively compact Borel sets satisfying

(1)

(i) $U_{n+1} \subsetneq U_n$ for all $n \in Z$ and $\lim_{n \to -\infty} \lambda(U_n) = \infty$;

(ii) $\lambda(U_n U_n^{-1}) < C\lambda(U_n)$, C constant, $n \in Z$;

(iii) For each $n \in Z$ there is an $l(n) \in Z$ such that $U_{l(n)} \supset U_n^{-1} U_n$ and $U_j \not\supset U_n^{-1} U_n$ if $j > l(n)$. And, there is a constant α such that $\lambda(U_{l(n)}) < \alpha\lambda(U_n)$ for all $n \in Z$.

For such an "M-sequence," we can prove [PT] the following theorem.

(2) COVERING THEOREM. *Let* $\mathfrak{U}=\{xU_n : x\in G,\ n\in Z\}$. *Suppose* $E\subset G$ *and* $\mathfrak{U}^\dagger\subset\mathfrak{U}$ *satisfy*

(i) $\lambda(EU_n)<\infty$ *for all* $n\in Z$;

(ii) *for each* $x\in E$, *there is an* n *such that* $xU_n\in\mathfrak{U}^\dagger$;

(iii) $\{n : xU_n\in\mathfrak{U}^\dagger$ *for some* $x\in E\}$ *is bounded below.*

Then, there are sequences $(x_k)_{k=1}^{\kappa}(1\leqq\kappa\leqq\infty)$ *in* E *and* $(n_k)_{k=1}^{\kappa}$ *in* Z *such that*

(iv) $\{x_kU_{n_k}\}_{k=1}^{\kappa}$ *is a pairwise disjoint family in* $\mathfrak{U}^\dagger$;

(v) $\lambda(E)\leqq C\sum_{k=1}^{\kappa}\lambda(U_{n_k})$.

The Covering Theorem yields the weak type estimate (3), below. For a locally integrable function f and a positive finite regular Borel measure μ, let

$$M_nf(x) = \frac{1}{\lambda(U_n)}\int_{xU_n} fd\lambda,\ n\in Z;\ Mf(x) = \sup\{M_nf(x) : n\in Z\}$$

$$M_n\mu(x) = \frac{\mu(xU_n)}{\lambda(U_n)},\ n\in Z;\ M\mu(x) = \sup\{M_n\mu(x) : n\in Z\}.$$

For a nonnegative function g and $t>0$, let $E_t[g]=\{x : g(x)>t\}$. If $1\leqq r<\infty$, $t>0$, and $f\in L_r$, then

$$\text{(3)}\qquad \begin{aligned} &\text{(i) } \lambda(E_t[Mf]) \leqq \frac{C}{t}\int_{E_{t/\alpha}[Mf]} fd\lambda.\\ &\text{(ii) } \lambda(E_t[M\mu]) \leqq \frac{C}{t}\,\mu(E_{t/\alpha}[M\mu]).\end{aligned}$$

It is surprising that the dependence on α appearing in (3) can be removed to give the following second weak type estimate.

$$\text{(4)}\qquad \lambda(E_t[Mf]) \leqq \frac{C}{(1-k)t}\int_{E_{kt}[f]} fd\lambda,\quad \text{all } k\in\,]0,1[.$$

With (3) and (4) in hand, classical methods (see [P]) are used to prove the following integral estimates.

(5) INTEGRAL ESTIMATES.

$$\text{(i) } f\in L_r^+ \Rightarrow \|Mf\|_r \leqq \frac{r}{r-1}\min[(Cr)^{1/r}, C\alpha^{r-1}]\|f\|_r.$$

(ii) $\{f \in L_1^+, k \in]0,1[, s \in]0,1[, E\ \lambda\text{-measurable}\} \Rightarrow$

$$\text{(a)} \int_E [Mf]d\lambda \leqq \frac{\lambda(E)}{k} + \frac{C}{1-k}\int_G f[\log^+ f]d\lambda;$$

$$\text{(b)} \int_E [Mf]^s d\lambda \leqq C^s \frac{\lambda(E)^{1-s}}{1-s}\left[\int_G f d\lambda\right]^s.$$

In [**PT**], relationships of the above results to maximal theorems of Calderón [**C**] and Smith [**S**] are also discussed.

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California Institute of Technology, Pasadena, California 91109 and
New Mexico State University, University Park, New Mexico 88001