SURGERY IN $M \times N$ WITH $\pi_1 M \neq 1$

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1. We announce here the answer, in part, to a question raised by Wall in §9 of [3], his basic paper on nonsimply connected surgery. To explain this, let X be a finite Poincaré complex of formal dimension m, and let ν be a vector bundle over X of the fiber homotopy type of the "Spivak normal fibration." In §3 of [3] Wall defines a cobordism group $\Omega_m(X, \nu)$ based on degree 1 maps $\phi: M \to X$ and framings of $T(M) \oplus \phi^*\nu$. In §5 (for m even) and §6 (for m odd) Wall defines a covariant functor L_m from finitely presented groups to abelian groups and a map $\theta: \Omega_m(X, \nu) \to L_m(\pi_1 X)$ which describes the obstruction to surgering ϕ: $\theta(\alpha) = 0$ if and only if α contains a simple homotopy equivalence $\phi: M \to X$. L_m and L_{m+4} are the same by definition. To give a geometric expression to this periodicity, in §9 Wall defines a pairing

$$L_m(\pi) \otimes \Omega_n \to L_{n+m}(\pi)$$

by associating, to N^n and the map $\phi: M \to X$, the product $\phi \times \mathrm{id}: M \times N^n \to X \times N^n$. This makes $L_*(\pi)$ into an Ω^*-module and Wall shows that the action of $[CP_2]$ is the periodicity identity $L_m = L_{m+4}$; Wall then conjectures that the action of $[N]$ depends only on the index $I(N)$. Here we show that this is true, at least for m odd and n even.

THEOREM 1. *For m odd and $n = 2r$, the pairing $L_m(\pi) \otimes \Omega_n \to L_{n+m}(\pi)$ sends $\alpha \otimes [N] \to I(N)\alpha$ for r even, $\alpha \otimes [N] \to 0$ for r odd.*

The case $m = 2k$ appears to be easier to handle, since the obstruction is the intersection form, which is just the $\otimes$-product of the form on M and the form on N, and is homologically defined. The self intersection form does introduce a complication, at least if k is odd. In any case we concentrate here on $m = 2k + 1$.

2. We freely use here terms and notation introduced by Wall in [3, §5, §6 mostly]. Throughout π will be a fixed finitely presented group and $\Lambda = \boldsymbol{Z}[\pi]$. Let (K, λ, μ) be a standard kernel, as in Wall's §5. So K is a free Λ-module (of finite dimension) and $K = S_1 \oplus S_2$ where S_1 has a specified basis $e_1, \cdots, e_\nu$ and S_2 has basis $f_1, \cdots, f_\nu$. λ is a $(-1)^k$-conjugate symmetric quadratic form on K (briefly,

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a $(-1)^k$-form), where the m of Theorem 1 is $m=2k+1$. The self intersection function $\mu:K\to\Lambda/\{\nu-(-1)^k\bar{\nu}\}$ is 0 on S_1 and S_2. λ is given by $\lambda(e_i, f_j)=(-1)^k\lambda(f_j, e_i)=\delta_{ij}$, and $\lambda=0$ on other base pairs.

Let Q be a unimodular bilinear integral-valued form on the free (finite rank) $\boldsymbol{Z}$-module H, $(-1)^r$-symmetric, $n=2r$, n as in Theorem 1. Then $K\otimes_Z H$ is also a Λ-module, so let $\lambda\otimes Q$, $\mu\otimes Q$ be the forms on it defined by

$$(\lambda\otimes Q)(x_1\otimes h_1, x_2\otimes h_2) = (-1)^{kr}\lambda(x_1, x_2)Q(h_1, h_2)$$
$$(\mu\otimes Q)(x\otimes h) = (-1)^{kr}\mu(x)Q(h, h).$$

LEMMA. *$K\otimes_Z H$ with $\lambda\otimes Q$, $\mu\otimes Q$ is a $(-1)^{k+r}$-kernel with subkernels $S_1\otimes_Z H$ and $S_2\otimes_Z H$.*

The only ambiguity in identifying $K\otimes_Z H$ with the standard kernel lies in the choice of basis of $S_1\otimes_Z H$; we choose $\{e_i\otimes h_j\}$ where $h_1, \cdots, h_p$ is a basis for H. The simple class of this basis does not depend on the basis $h_1, \cdots, h_p$, so we ignore this ambiguity. This lemma is proved using Lemma 5.3 of [3].

Now suppose $\alpha:K\to K$ is a simple isomorphism of the kernel K (preserving λ, μ). Then $\phi\otimes\mathrm{id}:K\otimes_Z H\to K\otimes_Z H$ is also a simple isomorphism.

THEOREM 2. *The map $\alpha\mapsto\alpha\otimes\mathrm{id}$ induces a homomorphism $\rho(Q)=L_m(\pi)\to L_{m+n}(\pi)$ which is $[\alpha]\to I(Q)[\alpha]$ if r is even $(n=2r)$ and is $[\alpha]\mapsto 0$ if r is odd, where $I(Q)=$ index (Q).*

We first observe that $\alpha\mapsto\alpha\otimes\mathrm{id}$ induces a map on the stable group, $\tilde{\rho}(Q):SU(\Lambda)\to L_{m+n}(\pi)$, and this map is additive: $\tilde{\rho}(Q_1\oplus Q_2)=\tilde{\rho}(Q_1)+\tilde{\rho}(Q_2)$. Furthermore, if Q_1 is equivalent to Q_2, $\tilde{\rho}(Q_1)=\tilde{\rho}(Q_2)$. For r even this reduces the proof to the verification of two special cases: first, $H\cong\boldsymbol{Z}$ and Q has matrix (1) and second, $H\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$ and Q has matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In the first case $K\otimes_Z H\cong K$ and $\alpha\otimes\mathrm{id}$ is carried to α, so $\tilde{\rho}(Q)(\alpha)=\alpha$. In the second case, let h_1, h_2 be the basis of the matrix representation. Then the submodule with basis

$$\{e_i\otimes h_1+e_i\otimes h_2, f_j\otimes h_1+f_j\otimes h_2\}\qquad 1\leqq i, j\leqq\nu$$

is a subkernel, although not one of the $S_i\otimes_Z H$—one checks immediately that $\lambda\otimes Q$, $\mu\otimes Q$ vanish on it. This subkernel is invariant under $\alpha\otimes\mathrm{id}$, and it follows that $[\alpha\otimes\mathrm{id}]=0$ in $L_{n+m}(\pi)$, which verifies the second case.

For r odd, it is enough to check $\tilde{\rho}(Q)=0$ for $H\cong Z\oplus Z$, Q with matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here we observe that

$$\{e_i \otimes h_1, f_j \otimes h_1\} \qquad 1 \leqq i, j \leqq \nu$$

is an invariant subkernel, so again $\tilde{\rho}(Q)=0$.

3. Here we follow Wall [3, §6]. Suppose X is a connected finite Poincaré complex of formal dimension $m=(2k+1)\geqq 5$ and ν is a Spivak normal bundle, as in §1. Let (M, ϕ, F) represent an element of $\Omega_m(X, \nu)$, so we may suppose $\phi: M\rightarrow X$ has degree 1 and F is a stable framing of $T(M)\oplus\phi^*\nu$. We omit mention of a boundary, which is to be mapped by a simple homotopy equivalence throughout. We can suppose ϕ is k-connected and U is the union of the images of disjoint embeddings of $S^k\times D^{k+1}$, each assigned a path connecting it to the base point, representing generators of $\pi_{k+1}(\phi)=K_k(M, \Lambda)$, $\Lambda=Z[\pi_1 X]$. We may suppose that X has only one m-cell D^m so there is a Poincaré pair (X_0, S^{m-1}), $X_0\cup D^m=X$ and ϕ induces a map of Poincaré triads, $M_0=M-\text{Int } U$,

$$\phi: \quad (M, M_0, U)\rightarrow(X; X_0, D^m).$$

With this set-up, Wall shows that the obstruction to surgering ϕ to a simple homotopy equivalence is represented by the pair of subkernels of the kernel $K_k(\partial U)$ which are the images in

$$K_{k+1}(M_0, \partial U)\rightarrow K_k(\partial U)\leftarrow K_{k+1}(U, \partial U).$$

The homology groups K_q are with coefficients Λ. In other words, if $\alpha\in SU(\Lambda)$ is an automorphism of $K_k(\partial U)$ which carries the right-hand subkernel to the left, then $[\alpha]\in L_m(\pi_1 X)$ is 0 iff (M, ϕ, F) is cobordant to (M', ϕ', F'), ϕ' a simple homotopy equivalence. In §7 this is generalised by Wall in such a way that M_0, U appear in completely symmetric roles.

Now let N be a smooth n-dimensional manifold, and $\pi_1(N)\cong 1$. Then we have $\phi\times\text{id}: M\times N\rightarrow X\times N$, and if ν_N is a normal bundle of N, then there is a unique framing F_N of $T(M\times N)\oplus(\phi\times\text{id})^*(\nu\times\nu_N)$, and we have an element

$$(M\times N, \phi\times\text{id}, F_N)\in\Omega_{m+n}(X\times N, \nu\times\nu_N).$$

This determines the pairing $L_m(\pi)\otimes\Omega_n\to L_{n+m}(\pi)$ mentioned above, defined in §9 of [3].

THEOREM 3. *For $m=2k+1$, $n=2r$, the surgery obstruction of $(M\times N, \phi\times\mathrm{id}, F_N)$ is represented by the subkernel pair*

$$K_{k+1}(M_0, \partial U)\otimes_Z H_r(N) \to K_k(\partial U)\otimes_Z H_r(N) \leftarrow K_{K+1}(U, \partial U)\otimes_Z H_r(N)$$

with intersection forms $\lambda\otimes Q$, $\mu\otimes Q$ on the kernel $K_k(\partial U)\otimes_Z H_r(N)$, where λ, μ are the forms on $K_k(\partial U)$ and Q is the intersection pairing on $H_r(N)$ (modulo torsion throughout).

Theorem 1 now follows from Theorem 2 and Theorem 3, together with the fact that every element of $L_m(\pi, X)$ can be realised as an obstruction.

We can divide $M\times N$ into two pieces, $M\times N=M_0\times N\cup U\times N$, and this defines a pair of subkernels exactly as stated in the theorem:

$$\begin{array}{ccccc} K_{k+r+1}(M_0\times N, \partial U\times N) & \to & K_{k+r}(\partial U\times N) & \leftarrow & K_{k+r+1}(U\times N, \partial U\times N) \\ \wr\| & & \wr\| & & \wr\| \\ K_{k+1}(M_0, \partial U)\otimes_Z H_r(N) & \to & K_k(\partial U)\otimes_Z H_r(N) & \leftarrow & K_{k+1}(U, \partial U)\otimes_Z H_r(N). \end{array}$$

If we could identify this subkernel pair as the obstruction we would be done, and by Lemma 7.2 of [3] we could do so if ϕ were $k+r$-connected on $M_0\times N$, $\partial U\times N$ and $U\times N$ and $k+r+1$-connected on $(M_0\times N, \partial U\times N)$, $(U\times N, \partial U\times N)$, but of course these conditions are not satisfied in general. So the proof consists of showing that these spaces can be surgered in such a way as to make them highly connected without disturbing the subkernel pair or the intersection form. Particular care is required to insure that the basis class is not changed either. The details of this proof will be given elsewhere.

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