CATEGORIES OF V-SETS

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Let V be a partially ordered set. Then a *V-set* is a function $A:X \to V$ from a set X to V. V is the set of *values* for A, and X is the *carrier* of A. If B: $Y \to V$ is another V-set, a *morphism* f: $A \to B$ is a function $\bar{f}$: $X \to Y$ such that $A(x) \leqq B(\bar{f}(x))$ for each $x \in X$. The category of all V-sets is denoted $\mathcal{S}(V)$. The *carrier functor* K: $\mathcal{S}(V) \to \mathcal{S}$ assigns X to $A:X \to V$ and $\bar{f}$: $X \to Y$ to f: $A \to B$, where $\mathcal{S}$ is the category of sets. See [2].

If V has one point, $\mathcal{S}(V) = \mathcal{S}$. If $V = \{0, 1\}$, where $0 < 1$, $\mathcal{S}(V)$ is the category of pairs (X, A) of sets, where $A \subseteq X$. If V is the closed unit interval, $\mathcal{S}(V)$ is the category of "fuzzy sets", as used by Zadeh and others [1], [5] for problems of pattern recognition and systems theory. When V is a Boolean algebra, V-sets are Boolean-valued sets, as used by Scott and Solovay for independence results in set theory (however, their notion of morphism is different).

If V is complete, $\mathcal{S}(V)$ is a pleasant category satisfying all Lawvere's axioms [3] for $\mathcal{S}$ except choice, modulo some substitutions of the V-set with carrier 1 and value 0 for the terminal object. In particular,

THEOREM 1. *If V is complete, $\mathcal{S}(V)$ is complete and cocomplete, has an exponential (i.e., a coadjoint to product) and a "Dedekind-Pierce object" (i.e., an object which looks like the set of integers; see* [3]).

Let Poc denote the category of partially ordered classes, and let $\mathcal{L}$ be a subcategory of Poc. Then a category $\mathcal{C}$ is *$\mathcal{L}$-ordered* if the *power function* $\mathcal{P}$: $|\mathcal{C}| \to$Poc factors through $\mathcal{L}$, where $\mathcal{P}(A)$ is the class of all equivalence classes of monics with codomain A ($f \equiv g$ if $\exists$ an isomorphism h such that $fh = g$). Denote the image of $A \xrightarrow{f} B$ by $f(A)$, and the image of the composite $A' \xrightarrow{i} A \xrightarrow{f} B$, where i is monic, by $f(A')$. Then $\mathcal{C}$ has *associative images* if it has images such that $f(g(A)) = (fg)(A)$, whenever $A \xrightarrow{g} B \xrightarrow{f} C$. $\mathcal{P}$ can be construed as a functor when $\mathcal{C}$ has associative images. Let CL denote the category of complete lattices, and call a category C_1 if a coproduct of monics is always monic.

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THEOREM 2. *A CL-ordered category with associative images has equalizers, inverse images, unions, intersections, and epic images. If it has coproducts, it is* C_1.

An object P in a category $\mathcal{C}$ is *monic* if every arrow $P \to A$ is monic, and is further *atomic* if every $P \to A$ is atomic in $\mathcal{P}(A)$. P is *good* if the functor $[P, \ ]: \mathcal{C} \to \mathcal{S}$ is noninitial preserving. A union $\bigcup_i A_i$ in $\mathcal{C}$ is *disjoint* if $i \neq j \Rightarrow A_i \cap A_j = \emptyset$, where $\emptyset$ is the initial object. Let CDL be the category of completely distributive lattices, i.e., complete lattices satisfying the law $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$. Such lattices V have *pseudo-complement* operators $*$: $V \to V$ defined by $a^* = \bigvee \{ b \mid a \wedge b = 0 \}$. Call $V \in |\mathrm{CDL}|$ *disjointed* if for each pair x, y of unequal atoms, $x^* \vee y^* = I$, the maximal element of V, and call $\mathcal{C}$ *disjointedly CDL-ordered* if each $\mathcal{P}(A) \in |\mathrm{CDL}|$ is disjointed.

THEOREM 3. *A category* $\mathcal{C}$ *is equivalent to* $\mathcal{S}(V)$ *for some* $V \in |CDL|$ *if and only if:*

(1) $\mathcal{C}$ *has an atomic monic good projective generator* P;

(2) $\mathcal{C}$ *has initial and terminal objects,* $\emptyset$ *and* I, *respectively*;

(3) $\mathcal{C}$ *has coproducts, which are disjoint unions; and conversely, each disjoint union in* $\mathcal{C}$ *is a coproduct in* $\mathcal{C}$;

(4) $\mathcal{C}$ *has associative images;*

(5) $\mathcal{C}$ *is disjointedly CDL-ordered; and*

(6) $P \amalg P$ *is not isomorphic to* P.

The Axioms (1)–(6) are easily verified for $\mathcal{S}(V)$, $V \in |\mathrm{CDL}|$. We now sketch the converse, which (surprisingly) makes no use of adjoint functors. Essential use is made of Theorem 2, via Axioms (4) and (5).

Call the elements of $[P, A]$ the *points* of A. We first show the one-pointed objects of $\mathcal{C}$ are the subobjects of I, except $\emptyset$; denote this lattice V. A calculation shows that each $A \in |\mathcal{C}|$ is a disjoint union $\bigcup_{x \in [P,A]} x^{**}$, so by Axiom (3), $A = \amalg_{x \in [P,A]} x^{**}$. These facts combine to show that each A is a subobject of $I^{[P,A]}$, the coproduct of I over the index set $[P, A]$. We then show the arrows f: $A \to B$ in $\mathcal{C}$ are in 1-1 correspondence with appropriate arrows $\bar{f}$: $[P, A] \to [P, B]$ in $\mathcal{S}$. The functor E: $\mathcal{C} \to \mathcal{S}(V)$ defined by $K(E(A)) = [P, A]$, $E(A)(x) = x^{**} \in V$, and $E(f) = [P, f]$, is then shown to be full, faithful, and representative.

The addition to Axioms (1)–(6) of either the categorical axiom of choice, or the condition $I = P$, yields a characterization of $\mathcal{S}$. For finite distributive lattices V, categories of V-sets with *finite* carrier are similarly characterized by all elementary axioms.

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Line 3. $\tilde{\mathcal{O}}^p/\tilde{f}^p$ should read $\tilde{\mathcal{O}}^p/f\tilde{\mathcal{O}}^p$.

Line 9. $\alpha(t^i_m) = s^i_m$ should read $\alpha(t^i_m) = s^i_m - \sum_{q=1}^{m-1} a_q s_q^{(i)}$.

Line 10 from bottom. $\mathbf{C}^N$ should read $\mathbf{C}^n$.