CONFORMALITY AND ISOMETRY OF RIEMANNIAN MANIFOLDS TO SPHERES

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Let M^n be a Riemannian manifold of dimension $n \geqq 2$ and class C^3, (g_{ij}) the symmetric matrix of the positive definite metric of M^n, and (g^{ij}) the inverse matrix of (g_{ij}), and denote by ∇_i, R_{hijk}, $R_{ij} = R^k_{ijk}$ and $R = g^{ij}R_{ij}$ the operator of covariant differentiation with respect to g_{ij}, the Riemann tensor, the Ricci tensor and the scalar curvature of M^n respectively. Throughout this paper all Latin indices take the values $1, \cdots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively, and that repeated indices imply summation.

Let v be a vector field defining an infinitesimal conformal transformation on M^n. Denote by the same symbol v the 1-form corresponding to the vector field v by the duality defined by the metric of M^n, and by L_v the operator of the infinitesimal transformation v. Then we have

$$L_v g_{ij} = \nabla_i v_j + \nabla_j v_i = 2\rho g_{ij}. \tag{1.1}$$

The infinitesimal transformation v is said to be homothetic or an infinitesimal isometry according as the scalar function ρ is constant or zero. We also denote by $L_{d\rho}$ the operator of the infinitesimal transformation generated by the vector field ρ^i defined by

$$\rho^i = g^{ij}\rho_j, \qquad \rho_j = \nabla_j \rho. \tag{1.2}$$

Let $\xi_{i_1 \cdots i_p}$ and $\eta_{i_1 \cdots i_p}$ be two tensor fields of the same order $p \leqq n$ on a compact orientable manifold M^n. Then the local and global scalar products $\langle \xi, \eta \rangle$ and (ξ, η) of the tensor fields ξ and η are defined by

$$\langle \xi, \eta \rangle = \frac{1}{p!} \xi^{i_1 \cdots i_p} \eta_{i_1 \cdots i_p}, \tag{1.3}$$

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$$(\xi, \eta) = \int_{M^n} \langle \xi, \eta \rangle dV, \tag{1.4}$$

where dV is the element of volume of the manifold M^n at a point.

In the last decade or so various authors have studied the conditions for a Riemannian manifold M^n of dimension $n>2$ with constant scalar curvature R to be either conformal or isometric to an n-sphere. Very recently Yano, Obata, Hsiung and Mugridge (see [6], [4], [2]) have been able to extend some of the above-mentioned results by replacing the constancy of R by $L_u R=0$, where u is a certain vector field on M^n. The purpose of this paper is to continue their work by establishing the following theorems.

To begin we denote by (C) the following condition:

(C) A compact Riemannian manifold M^n of dimension $n>2$ admits an infinitesimal nonisometric conformal transformation v satisfying (1.1) with $\rho \neq 0$ such that $L_v R=0$.

THEOREM I. *An orientable M^n is conformal to an n-sphere if it satisfies condition* (C) *and*

$$\left(\rho_i \rho^i - \frac{1}{n-1} R\rho^2, R\right) \geqq 0, \tag{1.5}$$

$$L_v\left(a^2 A + \frac{c - 4a^2}{n-2} B\right) = 0, \tag{1.6}$$

where A and B are defined by

$$A = R^{hijk} R_{hijk}, \qquad B = R^{ij} R_{ij}, \tag{1.7}$$

and a, c are constant such that

$$c \equiv 4a^2 + (n-2)\left[2a \sum_{i=1}^{4} b_i + \left(\sum_{i=1}^{6} (-1)^{i-1} b_i\right)^2 - 2(b_1 b_3 + b_2 b_4 - b_5 b_6) + (n-1) \sum_{i=1}^{6} b_i^2\right] > 0, \tag{1.8}$$

b's being any constants.

For the case $a \neq 0$, $c - 4a^2 = 0$ and the case $a=0$, $c-4a^2 \neq 0$, Theorem I is due to Yano [4].

THEOREM II. *A manifold M^n is conformal to an n-sphere, if it*

[2] An elementary calculation shows that $c \geqq 0$ where equality holds if and only if $b_1 = \cdots = b_4$, $b_5 = b_6 = 0$, $a = -(n-2)b_1$.

satisfies condition (C) *and any one of the following three sets of conditions:*

$$(1.9) \qquad \nabla_i\nabla_j(Rf) = R\rho g_{ij} \qquad (f \text{ is a scalar function}),$$

$$(1.10) \qquad Q\,d\rho = \frac{2}{n}\,d(R\rho), \qquad \nabla_i\nabla_j(R\rho) = R\nabla_i\nabla_j\rho,$$

$$(1.11) \qquad L_vR_{ij} = \alpha g_{ij} \qquad (\alpha \text{ is a scalar function}),$$

where Q is the operator of Ricci defined by, for any vector field u on M^n,

$$(1.12) \qquad Q: u_i \to 2R_{ij}u^j.$$

For constant R, conditions (1.10) and (1.11) in Theorem II will lead to the conclusion that M^n is isometric to an n-sphere of radius $(n(n-1)/R)^{1/2}$; for this see [**5**].

THEOREM III. *A manifold M^n with constant R is isometric to an n-sphere of radius $(n(n-1)/R)^{1/2}$, if it satisfies conditions* (C) *and* (1.9).

Theorem III is due to Lichnerowicz [**3**] when condition (1.9) is replaced by the following one:
(1.13) v is the gradient of a scalar function f, i.e., $v_i = \nabla_i f$.

For constant R, it is easily seen that condition (1.13) is a special case of condition (1.9). In fact, in this case by using (1.2) condition (1.9) becomes $\nabla_i v_j + \nabla_j v_i = 2\nabla_i\nabla_j f$, which is satisfied by $v_i = \nabla_i f + u_i$ where u_i is any vector field generating an infinitesimal isometry.

THEOREM IV. *A manifold M^n is isometric to an n-sphere, if it satisfies condition* (C), $L_{d\rho}R = 0$, *and*

$$(1.14) \qquad A^aB^b = c = \text{const},$$

$$(1.15) \qquad c\left(\frac{2a}{A} + \frac{(n-1)b}{B}\right) = \frac{2^a(a+b)R^{2(a+b-1)}}{n^{a+b-1}(n-1)^{a-1}},$$

where A, B are given by (1.7), *and a, b are nonnegative integers and not both zero.*

For constant R, Theorem IV is due to Lichnerowicz [**3**] for $a=0$, $b=1$ and due to Hsiung [**1**] for general a and b.

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