ON THE AVERAGE ORDER OF IDEAL FUNCTIONS AND OTHER ARITHMETICAL FUNCTIONS

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ABSTRACT. We consider a large class of arithmetical functions generated by Dirichlet series satisfying a functional equation with gamma factors. We state a general O-theorem for the average order of these arithmetical functions and apply the result to ideal functions of algebraic number fields.

Landau [4] and Chandrasekharan and Narasimhan [3] have proved O-theorems for the average order of a large class of arithmetical functions. The method of proof uses finite differences and is due to Landau. Often, it is desired to have an O-theorem where the error term is a function of a certain parameter, which is the discriminant, for example, in the case of an algebraic number field. We state here a general O-theorem of this type. The method of proof is a slight modification of Landau's mentioned above.

We briefly indicate the arithmetical functions under consideration. For a more complete description see [3].

Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers not identically zero. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive numbers tending to ∞. Put $s=\sigma+it$ with σ and t both real. We assume that

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s};$$

each converge in some half-plane and satisfy the functional equation

$$\Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s),$$

where r is real and

$$\Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu),$$

where $\alpha_\nu > 0$ and β_ν is complex, $\nu = 1, \cdots, N$.

In the sequel A always denotes a positive number not necessarily the same with each occurrence.

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For $q \geqq 0$, let

$$A_q(x) = \frac{1}{\Gamma(q+1)} \sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^q.$$

Let

$$Q(x) = \frac{1}{2\pi i} \int_C \frac{\phi(s) x^s}{s} ds,$$

where C is a cycle encircling all of the integrand's singularities. We shall assume that $\lambda_n = \lambda \lambda_n^*$ and $\mu_n = \lambda \mu_n^*$ where $\lambda > 0$ is a constant for the particular pair of Dirichlet series φ and ψ, and where λ_n^* and μ_n^* are not functions of λ. E.g., if we consider the zeta-function of an algebraic number field K, then $\lambda = d^{-1/2}$, where d is the modulus of the discriminant of K. Define

$$A_q^*(x, \lambda) = A_q(\lambda x) = \frac{\lambda^q}{\Gamma(q+1)} \sum_{\lambda_n^* \leq x} a(n)(x - \lambda_n^*)^q,$$

$$Q^*(x, \lambda) = Q(\lambda x),$$

and the error term

$$P^*(x, \lambda) = A_0^*(x, \lambda) - Q^*(x, \lambda).$$

Let σ_a^* denote the abscissa of absolute convergence of ψ. From [3, p. 111], $\sigma_a^* \geqq \frac{1}{2}r + 1/(4\alpha)$, where

$$\alpha = \sum_{\nu=1}^{N} \alpha_\nu. \tag{1}$$

The starting point for our investigation is an identity of Chandrasekharan and Narasimhan [3, p. 99]. If m is a sufficiently large positive integer and $x > 0$,

$$A_m(x) - S_m(x) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^{r+m}} I_m(\mu_n x), \tag{2}$$

where $S_m(x)$ arises from the singularities of $\Gamma(s)\varphi(s)$ and $d^m S_m(x)/dx^m = Q(x)$, and where

$$I_m(x) = \frac{1}{2\pi i} \int_{c_m - i\infty}^{c_m + i\infty} \frac{\Gamma(r-s)\Delta(s)}{\Gamma(m+1+r-s)\Delta(r-s)} x^{r+m-s} ds,$$

where $c_m = (\alpha r + m)/2\alpha - \epsilon$, $0 < \epsilon < 1/4\alpha$. We choose ϵ so that the path of integration contains no poles of the integrand.

We now state the

THEOREM. *Suppose that there is a positive integer m such that* (2) *holds,* $r/2+1/4\alpha+m/2\alpha>\sigma_a^*$, *and the integrand of* $I_m(x)$ *has no poles for* $c_0\leqq\sigma\leqq c_m$. *Suppose that the singularities of* φ *(if any) are at most poles (finite in number). Assume that there exist real constants* a, b, c, d, a', b', c' *and* d' *and a function* $f(\lambda)$ *such that*

$$A_0^*(x,\lambda),\ \sum_{\mu_n^*\leqq x} |b(n)| = O(x^a\lambda^b \log^c x\,|\log\lambda|^d)$$

and

$$Q^*(x,\lambda) = O(x^{a'}\lambda^{b'}\log^{c'}x\,|\log\lambda|^{d'}) + f(\lambda),$$

uniformly as x tends to ∞ *and* λ *tends to* 0. *Let*

$$\rho = \frac{b-b'+r-2a}{2\alpha a - r\alpha + 1/2} \qquad (2\alpha a - r\alpha + \tfrac{1}{2} \neq 0),$$

and $z=(x^\eta\lambda^{1/\alpha-\rho})^{2\alpha}$, *where* $\eta\geqq 0$ *and* α *is given by* (1). *Define*

$$E(x,\lambda) = x^{a'-1/2\alpha-\eta}\lambda^{b'+\rho}\log^{c'}x\,|\log\lambda|^{d'} + x^{r/2-1/4\alpha+\eta(2\alpha a-r\alpha-\frac{1}{2})}\lambda^{b'+\rho}\log^c z\,|\log\lambda|^d.$$

Assume that, for $x^{1/2\alpha+\eta}\leqq A\lambda^\rho$,

$$x^a\lambda^b\log^c x\,|\log\lambda|^d = O\{E(x,\lambda)\},$$

and that, for $x^{1/2\alpha+\eta}\geqq A\lambda^\rho$,

$$f(\lambda) = O\{E(x,\lambda)\},$$

uniformly as x tends to ∞ *and* λ *tends to* 0. *Then, if* $\sigma_a^*>r/2+1/4\alpha$,

$$P^*(x,\lambda)+f(\lambda) = O\Bigg(\sum_{x<\lambda_n^*\leqq x+O(x^{1-1/2\alpha-\eta}\lambda^\rho)} |a(n)|\Bigg) + O\{E(x,\lambda)\},$$

uniformly as x tends to ∞ *and* λ *tends to* 0. *Furthermore, if* $a(n)\geqq 0$,

$$P^*(x,\lambda)+f(\lambda) = O\{E(x,\lambda)\}.$$

If $\sigma_a^*=r/2+1/4\alpha$, we have the same results as above, except that an additional factor of $\log z$ must be placed in the second term defining $E(x,\lambda)$.

EXAMPLE. Let K be an algebraic number field of degree $n=r_1+2r_2$, where r_1 is the number of real conjugates and $2r_2$ the number of imaginary conjugates in K. The Dedekind zeta-function for K is defined by

$$\zeta_K(s) = \sum_{\nu=1}^\infty F(\nu)\nu^{-s}, \qquad \sigma>1,$$

where $F(\nu)$ is the number of nonzero, integral ideals of norm ν in K. Furthermore, $\zeta_K(s)$ satisfies the functional equation

$$\xi(s) = \xi(1-s),$$

where $\xi(s) = B^s d^{s/2s}\Gamma^{r_1}(\frac{1}{2}s)\Gamma^{r_2}(s)\zeta_K(s)$, where B is a constant depending only on n, and d is the modulus of the discriminant of K. In the notation of our theorem we have $\alpha = \frac{1}{2}n$ and $\lambda = d^{-1/2}$. Also, it is well known that

$$Q^*(x, d^{-1/2}) = Q^*(x, d^{-1/2}, n) = c_1 hRd^{-1/2}x + \zeta_K(0),$$

where h is the class number of K, R the regulator of K, and c_1 a constant depending only on n. If K is an imaginary quadratic field, $\zeta_K(0) = c_2 h$, where c_2 does not depend upon d; otherwise, $\zeta_K(0) = 0$. From [5, p. 481] we have

$$hR \leqq Ad^{1/2}\log^{n-1} d,$$

where A depends only on n. Thus,

$$|Q^*(x, d^{-1/2}, n)| \leqq Ax\log^{n-1}d + |\zeta_K(0)|,$$

where $|\zeta_K(0)| \leqq Ad^{1/2}\log d$ if K is an imaginary quadratic field. Also, from [5, p. 482],

$$\sum_{\nu \leqq x} F(\nu) \leqq Ax\log^{n-1} d,$$

where A depends only on n and not on x or d. In our theorem we can take $m = n$. Also, $\rho = -2/(n+1)$. Choose $\eta = (n-1)/\{n(n+1)\}$. Thus,

$$E(x, d^{-1/2}) = E(x, d^{-1/2}, n) = 2x^{(n-1)/(n+1)}d^{1/(n+1)}\log^{n-1} d.$$

For $x^{2/(n+1)} \leqq Ad^{1/(n+1)}$,

$$x\log^{n-1} d = x^{(n-1)/(n+1)}x^{2/(n+1)}\log^{n-1} d \leqq AE(x, d^{-1/2}, n).$$

For an imaginary quadratic field and $x^{2/3} \geqq Ad^{1/3}$,

$$|\zeta_K(0)| \leqq Ad^{1/2}\log d = Ad^{1/3}d^{1/6}\log d \leqq AE(x, d^{-1/2}, 2).$$

Thus, all the hypotheses of our theorem are satisfied, and we conclude that

$$\sum_{\nu \leqq x} F(\nu) - c_1 hRd^{-1/2}x = O(x^{(n-1)/(n+1)}d^{1/(n+1)}\log^{n-1} d). \tag{3}$$

This problem has been considered by several authors. Ayoub [1], for an imaginary quadratic field, showed that the left side of (3) is

$O(x^{1/3+\epsilon}d^{1/3+\epsilon})+O(x^{\epsilon}d^{1/2+\epsilon})$ for every $\epsilon>0$. Fögels has considered the problem and some generalizations. (See [2] and other papers of the author cited there.) The best results, however, were previously achieved by Landau [5] who showed that the left side of (3) is $O(x^{(n-1)/(n+1)}d^{1/(n+1)}\log^n d)$. Our result is better than Landau's by a factor of $\log d$. However, an examination of Landau's proof shows that his proof really yields the slightly better result that we give.

Our theorem also yields results for L-series of algebraic number fields.

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