to me that the more standard terminology "Poincaré metric" would have been more suitable.

(c) Problem 1 on p. 131 has been solved by Mark Green, who has obtained best possible degeneracy results for holomorphic maps of C^k into P_n which omit a certain number of hyperplanes in general position.

(d) The proof of Theorem 3.1 on p. 83 is due to Grauert and Reckziegel, and it was Mrs. Kwack who recognized that their argument had wider implications than those which they gave.

(e) Finally, I should like to offer a clarification concerning the result mentioned in example 2 on p. 94. Let D be a bounded symmetric domain in C^n, Γ an arithmetic group of automorphisms of D, and $M = D/\Gamma$ the quotient. There are two compactifications N_1 and N_2 of M due respectively to Baily-Borel-Satake and Pyatetzki-Shapiro. For the first, there is a fairly complicated description of the fundamental domain Ω_1 for Γ acting on D, and, using this, N_1 turns out to be a Hausdorff space which carries the structure of a complex-analytic variety. For the second, there is a much easier description of the fundamental domain Ω_2. It has just been recently proved by A. Borel that the natural mapping $h: N_1 \to N_2$ is a homeomorphism, so that the compactifications coincide. The extension theorem of Borel is for $f: D^* \to N_1$, and the proof is difficult because of the complicated nature of Ω_1. The Kobayashi-Ochiai result is for $f: D^* \to N_2$, and yields Borel's theorem only by using the identification $N_1 \tilde{\to} N_2$. The extension theorem of Borel is of great use in algebraic geometry, and has recently been used by P. Deligne to prove the Riemann hypothesis for algebraic K3 surfaces.

(f) Kiernan and Kobayashi have recently proved that a holomorphic mapping $f: D/\Gamma \to D'/\Gamma'$ between arithmetic quotients of bounded domains extends to a holomorphic mapping $f: N_2 \to N_2'$ between the compactifications. This result will appear in Ann. of Math.

(g) The answer to problem 2 appears in the papers by H. Cartan (Ann. École Norm Sup. **45** (1928), 256-346) and A. Bloch (Ann. École Norm Sup. **43** (1926), 309-362).

(h) The result in Kobayashi-Ochiai [2] originally appeared in P. Griffiths (Ann. of Math. **93** (1971), 439-458).

P. GRIFFITHS

Combinatorial identities by John Riordan. John Wiley and Sons, New York, 1968.

1. In the eighth book of his celebrated work, the geographer Strabo essays a detailed description of the whole of Greece. Strabo is well aware of the fact that his task is not completely straightforward, since it involves him in some highly conjectured identifications of sites mentioned by

Homer, who "speaks poetically, and not of things as they are now—." The intrepid scholar is resigned but undeterred, and the work proceeds according to what one presumes to be first century canons of orderly exposition. Problems of identification are briskly disposed of as they arise; nevertheless, one can occasionally detect a note of exasperation, even despair:

τῶν δ' ὑπὸ τοῦ ποιητοῦ λεγομένων

Ῥίπην τε Στρατίην τε καὶ ἠνεμόεσσαν Ἐνίσπην

εὑρεῖν τε χαλεπόν, καὶ εὑροῦσιν οὐδὲν ὄφελος διὰ τὴν ἐρημίαν.

(Strabo, 8.8.2)

In John Riordan's excellent and stimulating book *Combinatorial identities*, the note of despair is sounded in the preface:

"The object of this book is to present identities in mathematical settings that provide areas of order and coherence. My initial hope that some of this order and coherence would be acquired by the identities themselves now seems illusory. No useful criterion for ranking identities by interest and importance has emerged. The identities, both old and new, seem to lose whatever memorable character they have by the profusion that surrounds them. The reader is warned that he may not expect his identity of the moment, however fascinating it seems, to be listed and verified, although he may find clues to a region or an area of verification. The central fact developed is that identities are both inexhaustible and unpredictable; the age-old dream of putting order in this chaos is doomed to failure." (p. VII)

It is well that Riordan has written thus; as in the case of Strabo's geography, the extent to which the book succeeds is a matter for discussion among specialists, and Riordan himself is perhaps the pre-eminent practitioner of what one is tempted to call "formal" combinatorics. The question of formality will arise again below; at this point it is appropriate to remark that the author's judgement of his own work is rather too harsh.

2. The working definition of a combinatorial identity used in this book is one of satisfying catholicity. To quote again from the preface:

"The identities examined are not solely those for binomial coefficients; other combinatorial entities, like the Catalan, Fibonacci, and Stirling numbers, recur frequently and call for the generality of the title: *Combinatorial identities.* This is taken in the vague sense suitable to the growing state of combinatorial mathematics at the moment. In the first place, it encompasses all identities phrased in terms of recognized combinatorial entities, such as the old-fashioned permutations, combinations, variations, and partitions, and the numbers arising in their enumeration.

It leaves space for similar identities in the newer areas of trees, graphs, difference sets, block and code designs, and their esoteric extensions. Indeed, the term combinatorial identity may be regarded as designating any identity with combinatorial significance." (p. VIII)

The author then goes on to explain what the book is not about, and in so doing, outlines the contents of a further book, one that most combinatorialists would agree is urgently needed:

"One kind of identity, which receives only passing attention here, is that of combinatorial problems themselves, in the sense that two problems are the same if they have the same enumerator (enumerating generating function). An example, noted in my earlier book, is that of the several problems enumerated by the Stirling numbers. The mappings of such problems into each other are, of course, of great combinatorial interest; at the moment I know of relatively few mappings, and of no systematic examination, which is my excuse for almost ignoring them." (p. VIII)

It is clear from this that Riordan himself does not contemplate writing such a book, which is indeed a pity. With regard to the more elementary parts of the subject, the recent book by Claude Berge [1] provides some insight into such combinatorial connections. What is really needed, however, is a thorough-going treatment, comparable in scope to the book here under review. As of this writing, the prospects for the early appearance of such a work seem dim indeed.

3. *Combinatorial identities* is divided into six chapters, with the following titles: 1. Recurrence; 2. Inverse Relations I; 3. Inverse Relations II; 4. Generating Functions; 5. Partition Polynomials; 6. Operators. To the uninitiated researcher, looking for some formula that bears on his own special problem, these chapter headings are likely to be unhelpful. Indeed, at first glance the book resembles nothing so much as a maze; the reader is advised to employ some analogue of the proverbial colored thread so as not to lose his way. Many of the most striking and useful results appear, not in the text, but in the problems appended to each chapter. A beautiful formula, once discovered, may easily be lost irretrievably (if not written down), because the reason for its inclusion at that particular place may not be apparent. Unhappily, the index is far too sparse to be of real use. Furthermore, as is perhaps inevitable in a book so brimful of detail, along with the gold there is some diluting dross—a circumstance which considerably complicates the retrieval problem. One sometimes feels that Riordan's immense manipulative facility has got out of control, producing mathematical analogues of the Homeric cities cited by Strabo (above), which are "not only hard to find, but are of no use to anyone who finds them, because they are deserted." As the author himself puts it in speaking of the innumerable pairs of mutually inverse expressions:

"It is clear that the stock of such inverse relations is inexhaustible, but unfortunately there is no rule of significance to reduce this bounty to our competence." (p. 100)

How, then, is one to make use of this book? The first step is to put aside the notion that it is a sort of dictionary, a glossary to be mechanically consulted; on the contrary, its contents "nous prouvent que nous n'identifierions pas les objects si nous ne faisons pas intervenir le raisonnement" (Proust). *Combinatorial identities* is, in fact, a book that must be *read*, from cover to cover, and several times. In this way the reader will not only discover where things are (and why they are there), but will also come to perceive the inner logic of the development—quite strict in its fashion, at least within the confines of a given chapter (exception should be made of the chapter on Partition Polynomials, which is something of a hodge-podge).

The effect of such a reading is to produce in the reader a sense of confidence, even power, and to impart a zest for tackling the—frequently tedious—details which are the inescapable concomitant of "hard" combinatorics. Alas, this confidence may well turn out to be unjustified. Consider the following example:

Define a set of integers $P_{n,k}$ by means of the recurrence:

$$P_{n,k} = k^2\{P_{n-1,k-1} + P_{n-1,k}\}, \qquad P_{n,1} = 1.$$

Anyone familiar with *Combinatorial identities* will immediately connect the $P_{n,k}$ with the central factorial numbers $T(n, k)$ of Chapter 6:

$$P_{n,k} = (k!)^2 T(2n, 2k).$$

This identification will lead to a host of interesting relations, but what is one to do with the following simple identity:

$$\text{(I)} \qquad (n + 1)\sum(-1)^{m+1}\binom{2m}{m}2^{-2m}P_{n,m} = \sum(-1)^{m+1}P_{n,m}?$$

To make a long story short, nothing in the book turns out to be of immediate help in establishing (I). Actually, Riordan himself has verified the identity, but only by resort to highly indirect methods (see the forthcoming paper by Riordan and Stein: *Proof of a conjecture on Genocchi numbers*).

Many examples of this sort could be adduced; one hopes, nevertheless, that they will prove to be exceptional.

4. It is not the purpose of this review to analyze the contents of *Combinatorial identities* in detail; a few general remarks however, are in order. Its title notwithstanding, Chapter 1 is really a condensed course on how

to sum over products of binomial coefficients, the principal tool being the "Vandermonde Convolution":

$$\binom{n}{m} = \sum_{k=0} \binom{n-p}{n-k}\binom{p}{k}.$$

The theme is pursued further in Chapter 4 (see the section on "Multisection of Series" and that on "Cycles of Binomial Coefficients"). If one merely wished to tabulate results, one might perhaps adopt an alternative taxonomy (see Chapter 13 of Netto's *Kombinatorik* [2], to which, incidentally, no reference is made). The procedure adopted here, while not so convenient for the reader merely seeking ready reference, is far more illuminating than the standard expositions.

Chapters 2 and 3 on "Inverse Relations" are easily the best in the book. By an inverse relation is meant a pair of formulae like the following:

$$a_n = \sum_{k=n} \binom{k}{n} b_k, \qquad b_n = \sum_{k=n} (-1)^{k+n} \binom{k}{n} a_k \qquad (a_n, b_n \text{ unrestricted}).$$

From these humble beginnings there emerges a dazzling variety of pairs, some of which are of considerable intricacy. In addition to binomials, factorials, and assorted powers, one also finds Bernoulli and Euler numbers appearing as kernels in these relations. Multivariable pairs are also investigated, but not in depth, because, as the author says:

"$\cdots$ the possibilities open are overwhelming and there is less guide to their interest than the dim light previously available." (p. 113)

Chapter 4 on "Generating Functions" takes up the summation problem again, this time employing different tools. Of these perhaps the most interesting is what Riordan calls "multisection of series"—essentially a generalization of the hoary device of splitting a power series into its odd and even parts. In this chapter, as in the following two, heavy use is made of the symbolic or "umbral" calculus. This makes for hard reading (since many expressions have to be translated into ordinary notation in order to reveal their meaning), but the tool is without doubt indispensible. In this connection it should be remarked that much of the material in Chapters 4 and 6 could be abbreviated and unified by adopting the approach of Mullin and Rota [3]. The resultant gain in elegance, however, might well be offset by a loss of richness.

Chapter 5 is the least satisfactory in the book; it seems unlikely that it will prove to be of much help to the working combinatorialist. One gets the feeling that the author has somehow missed the point; perhaps it is only that he has chosen his examples less judiciously here than elsewhere. In contrast, Chapter 6 ("Operators") is of great interest, although, in this reviewer's opinion, it is rather too short. The presentation seems a little old-fashioned and, in some places, not far from the early exposition by

Steffensen [**4**] (to whom, unaccountably, no reference is made). Here one finds developments involving the Stirling numbers, the central factorial numbers, and other ubiquitous combinatorial sequences. To use this chapter properly, one must make frequent reference to the author's earlier book *Introduction to combinatorial analysis* [**5**]. In addition, it would be of help to consult the relevant chapters of Jordan's *Calculus of finite differences* [**6**]. Supplemented by these two books, the chapter is both stimulating and illuminating; without them, the reader may find the development a little arid and the examples lacking in motivation.

5. In his widely read *Introduction to combinatorial analysis*, John Riordan revealed himself to be primarily a formalist. His arguments as given there make only sparing use of "anschaulich" and diagrammatic methods and none at all of explicit mapping techniques. In this respect his approach is precisely opposite to that of, say, Claude Berge [**1**] (to be sure, the latter book is much more elementary than Riordan's, so that no direct comparison can be made). Indeed, Riordan is even more formal than that considerable formalist Major P. A. MacMahon [**7**], who was wont to present an occasional problem "in words" or with the help of clarifying diagrams. In *Combinatorial identities*, Riordan's formalist tendencies have attained their apotheosis; practically nowhere in the book is the combinatorial setting of a problem described. The whole development is, as it were, divorced from enumerative reality. The choice is, of course, deliberate; whether or not it constitutes a fault is a matter of taste. But the practical consequence is that the book is not selfcontained, that is, it cannot be "understood" by the nonspecialist without constant reference to some other, more general work on counting theory—preferably the author's own earlier book. This difficulty could have been largely alleviated by making the book some twenty pages longer, the contents of these pages to consist of combinatorial obiter dicta, distributed strategically throughout the text. The reviewer feels that such a course would have resulted in a book far more accessible to the interested but not necessarily dedicated reader than is the present one.

It is unlikely that John Riordan would take kindly to this suggestion. For him, the direct, intuitive approach to enumeration is "the hard way" (his phrase). For many others, whose formal powers are of a lesser order, it is "the easy way" and, indeed, sometimes the only way.

PAUL R. STEIN

REFERENCES

1. C. Berge, *Principes de combinatoire*, Dunod, Paris, 1968; English transl., Math. in Sci. and Engrg., vol. 72, Academic Press, New York, 1971. MR **38** # 5635; MR **42** # 5805.
2. E. Netto, *Kombinatorik*, reprint, Chelsea, New York.

3. R. Mullin and G.-C. Rota, "Theory of binomial enumeration," in *Graph theory and its applications*, edited by B. Harris, Academic Press, New York, 1970, pp. 167–213.
4. J. F. Steffensen, *Interpolation*, 2nd ed., Chelsea, New York, 1950. MR **12**, 164.
5. J. Riordan, *An introduction to combinatorial analysis*, Wiley Publ. in Math. Statist., Wiley, New York; Chapman & Hall, London, 1958. MR **20** # 3077.
6. C. Jordan, *Calculus of finite differences*, Chelsea, New York, 1950.
7. P. A. MacMahon, *Combinatory analysis*, 2 vols., Cambridge Univ. Press, Cambridge, 1915, 1916.

Current address: Los Alamos Scientific Laboratory, P. O. Box 1663, Los Alamos, New Mexico 89544