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ULTRAFILTERS AND ABELIAN GROUPS

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1. **Introduction.** Successful structure theorems for abelian p-groups, such as Ulm's theorem and its generalizations [2], [6], [10], [11], usually involve systems of cardinal invariants. Although such systems serve to classify large classes of groups, they do not appear to be adequate in general. This is due in part to the abundance of the groups, and in part to the various pathologies known to occur. In an attempt to remedy this situation we propose the use of invariants whose values are not cardinal numbers but rather ultrafilters on a countable set. Such invariants share many desirable properties with cardinals (for instance, they do not depend on the prime p involved), but provide much more flexibility than cardinals do. We present here a statement of some of our results. Details and further results will appear elsewhere.

A variation on our methods is applicable to the problem of classifying primitive rings with minimal left ideals. This will be touched on briefly in the conclusion, §5.

2. **Groups and filtered vector spaces.** Let G be a separable abelian p-group, i.e. one with no elements of infinite height. Then the socle $G[p]$ of G (the elements of order p) is a vector space over the field $\mathbf{F}_p$ of p elements. This vector space possesses a natural filtration (sometimes called a valuation [4]) induced by the height function on G, namely $G[p] \supseteq (pG)[p] \supseteq (p^2G)[p] \supseteq \cdots$. If G_1 and G_2 are isomorphic groups then their socles are isomorphic as filtered vector spaces. In this paper we concentrate on providing invariants for the socle, although some of the techniques can be applied to the groups directly.

Henceforth we make two assumptions which drastically simplify the situation: We assume that G is standard, i.e. that all of its Ulm invariants are ones, and furthermore that $\overline{G}/G$ is isomorphic to $\mathbf{Z}(p^\infty)$, where $\overline{G}$ is the torsion completion of G. (This is, in a sense, the simplest nontrivial case.) Then $(p^nG)[p]/(p^{n+1}G)[p]$ is one-dimensional for all n and $G[p]$ has codimension one in $\overline{G}[p]$, which can be regarded as the completion of $G[p]$ with respect to the filtration. Hence we are reduced to studying filtered vector spaces $S (= S_0 \supset S_1 \supset S_2 \supset \cdots)$ over $\mathbf{F}_p$ satisfying the following conditions: $\bigcap_n S_n = 0$, S_n/S_{n+1} is one-dimensional for all n, and $\overline{S}/S$ is one-dimensional (where $\overline{S}$ is the completion of S).

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We shall denote by $\mathcal{S}$ the collection of isomorphism classes of such S. (Here isomorphisms are of course to preserve filtrations.) It is known that any such S arises as $G[p]$ for some G.

3. **Filtered vector spaces and the second dual.** Let V be a vector space over $\mathbf{F}_p$ with a countably infinite basis $B = \{v_1, v_2, v_3, \ldots\}$. Then the second dual V^{**} of V is a vector space of dimension $2^{(2^{\aleph_0})}$ in which V is naturally embedded. Any endomorphism φ of V naturally induces an endomorphism φ^{**} of V^{**}, so the general linear group $GL(V)$, or any subgroup thereof, acts on the set V^{**}.

Let $UT(V)$ be the subgroup of $GL(V)$ consisting of those φ which preserve the (increasing) filtration $\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_3 \rangle \cdots$, i.e. such that $\varphi(v_n) \in \langle v_1, \ldots, v_n \rangle$ for all n. (UT stands for upper triangular). Then $UT(V)$ acts on V^{**} and we can consider the orbit space $V^{**}/UT(V)$. This contains the subset $V/UT(V) = \{\overline{0}, \overline{v}_1, \overline{v}_2, \ldots\}$ where $\overline{v}$ denotes the orbit of v. The rest, i.e. $(V^{**} - V)/UT(V)$, has cardinality $2^{(2^{\aleph_0})}$ and we have

There is a natural 1-1 *correspondence between* $(V^{**} - V)/UT(V)$ *and the collection* $\mathcal{S}$ *(of isomorphism classes of socles).*

4. **The second dual and ultrafilters.** Let $B = \{v_1, v_2, \ldots\}$ and let $\beta(B)$ denote the Stone-Čech compactification of B, i.e. the collection of ultrafilters $\mathcal{U}$ on B. Any such $\mathcal{U}$ induces an element $\lambda_{\mathcal{U}}$ of V^{**} as follows: if f is in V^* then f is constant almost everywhere (with respect to $\mathcal{U}$) on B, and we define $\lambda_{\mathcal{U}}(f)$ to be this constant. It is clear that different $\mathcal{U}$'s give different $\lambda_{\mathcal{U}}$'s, and in fact they give $\lambda_{\mathcal{U}}$'s lying in different $UT(V)$ orbits. Hence we have

There is a natural 1-1 *correspondence between* $\beta(B)$ *and a subset of* $V^{**}/UT(V)$.

Under this correspondence B corresponds to $V/UT(V)$ (omitting the element $\overline{0}$).

Now $\beta(B) - B$ has cardinality $2^{(2^{\aleph_0})}$ as does $(V^{**} - V)/UT(V)$. However it is far from true that all elements of $(V^{**} - V)/UT(V)$ are of the form $\overline{\lambda}_{\mathcal{U}}$. If we denote by L_1 ("first level") the elements of this form, by L_2 elements of the form $\overline{\lambda_{\mathcal{U}} + \lambda_{\mathcal{V}}}$, etc. then we have

The sequence $L_1 \subset L_2 \subset L_3 \subset \cdots$ *is strictly increasing and its union is a proper subset of* $(V^{**} - V)/UT(V)$.

5. **Conclusion.** The ultrafilters in $\beta(B) - B$ provide natural invariants for the subset L_1 of $(V^{**} - V)/UT(V)$ and hence for the corresponding subset of $\mathcal{S}$. In addition they establish a natural 1-1 correspondence between these subsets as the prime p varies.

The situation for the higher levels is less clear. Even for L_2, where pairs of ultrafilters ought to provide invariants, nasty problems (involving the Rudin-Keisler ordering [9] on $\beta(B)$) arise which we have only partially solved. Here the dependence on p remains open.

Beyond these questions lie the problems of extending the invariants to other classes of socles and to the groups themselves. It is worth pointing out, however, that isomorphism classes of groups G with $G/p^{\omega}G$ standard and torsion complete and $p^{\omega}G$ cyclic of order p correspond naturally to elements of S and hence of $(V^{**} - V)/UT(V)$. Therefore our results apply directly to one class of groups.

Finally, one can investigate instead the orbit space $V^{**}/GL(V)$, where $GL(V)$ denotes the full linear group on V. Here difficulties arise even sooner, i.e. at level one rather than two [5]. This orbit space can be used to provide invariants for certain classes of primitive rings with minimal left ideals via the dual space characterization [7].

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