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RANDOM REARRANGEMENTS OF FOURIER COEFFICIENTS

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In [2] and [3] Hardy and Littlewood characterized those sequences of numbers which for every variation of their arguments and arrangements are the Fourier coefficients of a function in L^p $(2 < p < \infty)$ and those sequences which for some variation of their arguments and arrangements are the Fourier coefficients of a function in L^p $(1 < p < 2)$. In [4] Paley and Zygmund characterized those sequences (c_n) such that for almost every choice of ± 1's, $(\pm c_n)$ is the sequence of Fourier coefficients of a function in L^p $(1 \leqslant p < \infty)$. Here we are interested in the following problem: Which sequences are for almost every variation of their arrangements the sequence of Fourier coefficients of a function in L^p? Of course it is necessary to make precise the phrase "almost every variation of their arrangements". Following Garsia [1] we consider a probability space X of "local" permutations of $\{1, 2, \ldots\}$: For $k = 0, 1, 2, \ldots$ let $S(2^k)$ be the symmetric group on the set $\{2^k, 2^k + 1, \ldots, 2^{k+1} - 1\}$. To each $\sigma_k \in S(2^k)$ we assign the probability $1/2^k$, and we let X be the product probability space $\Pi_{k=0}^{\infty} S(2^k)$. For $\sigma = (\sigma_0, \sigma_1, \ldots) \in X$ and a Fourier series of the form $\Sigma_{n=1}^{\infty} c_n e^{in\theta}$, we define

$$S(\sigma, \theta) = \sum_{k=0}^{\infty} \sum_{2^k \leqslant n < 2^{k+1}} c_{\sigma_k(n)} e^{in\theta}.$$

For $k = 0, 1, \ldots$ let

$$a_k = 2^{-k} \sum_{2^k \leqslant n < 2^{k+1}} c_n,$$

and let (*) stand for the statement that

$$\sum_{k=0}^{\infty} |a_k|^p 2^{k(p-1)} < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \sum_{2^k \leqslant n < 2^{k+1}} |c_n - a_k|^2 < \infty.$$

THEOREM 1. *For $2 < p < \infty$, the following are equivalent*:

(i) *for some $\sigma \in X$, $S(\sigma, \theta)$ is an L^p Fourier series*;

(ii) *$S(\sigma, \theta)$ is almost surely an L^p Fourier series*;

(iii) (*).

THEOREM 2. *For $1 < p < 2$, the following are equivalent*:

(i) *$S(\sigma, \theta)$ is an L^p Fourier series with positive probability*;

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(ii) *$S(\sigma, \theta)$ is an L^p Fourier series for every $\sigma \in X$;*

(iii) (*).

THEOREM 3. *If $a_k = 0$ for $k = 0, 1, 2, \ldots$, the following are equivalent:*

(i) *$S(\sigma, \theta)$ is a Fourier (-Stieltjes) series with positive probability;*

(ii) *it is almost sure that $S(\sigma, \theta)$ is the Fourier series of a function in every L^p $(1 \leqslant p < \infty)$.*

The proofs are based on a combinatorial estimation of the L^p norms of permuted Fourier series on finite groups and on classical results due to Hardy, Littlewood, Paley, and others. Analogous results obtain for Fourier series on the group $D^\infty = \Pi_{n=1}^\infty\{0, 1\}$.

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