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ON IDEALS OF SETS AND THE POWER SET OPERATION

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We present some inequalities involving cardinal powers. In most of the results we assume the existence of an ideal I satisfying a weak completeness condition.

For the remainder of this paper, *I will always denote an ideal over ω_1 containing all enumerable sets.* $F \subseteq P(\omega_1)$ is I-disjoint if $X \cap Y \in I$ for all distinct $X, Y \in F$; F is *almost disjoint* if $|X \cap Y| \leq \aleph_0$ for all distinct $X, Y \in F$. I is *λ-saturated* if $|F| < \lambda$ for every I-disjoint $F \subseteq P(\omega_1) - I$.

THEOREM 1. *Let I be σ-additive. If $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$, then for every $\lambda < 2^{\aleph_1}$ there is an almost disjoint $F \subseteq P(\omega_1) - I$ with $|F| = \lambda$. Moreover, if $2^{\aleph_1}$ is singular, we get such an F with $|F| = 2^{\aleph_1}$. Hence if $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$, then there exists no λ-saturated ideal for any $\lambda < 2^{\aleph_1}$.*

REMARK. In [1] the same assumption on $2^{\aleph_0}$ is used to obtain an almost disjoint F such that $|F| = 2^{\aleph_1}$. In [3] stronger assumptions on $2^{\aleph_0}$ are used to show that the ideal of nonstationary sets is not $\aleph_2$-saturated.

For $S \in P(\omega_1) - I$, W is an *I-partition* of S if W is a maximal I-disjoint family $\subseteq P(S) - I$. If W_0 and W_1 are I-partitions of S, then W_1 is a refinement of W_0 if every $X \in W_1$ is included in some $Y \in W_0$.

I is *precipitous* if for every $S \in P(\omega_1) - I$, and every sequence W_n $(n \in \omega)$ of I-partitions of S such that W_{n+1} is a refinement of W_n, there exists a sequence $X_n \in W_n$ such that $X_{n+1} \subseteq X_n$ and $\bigcap \{X_n : n \in \omega\} \neq 0$.

PROPOSITION. *If there is a precipitous I, then there is a σ-additive, normal, precipitous I. If I is normal and precipitous, then ω_1 is measurable in $L[I]$. If I is $\aleph_2$-saturated, then I is precipitous. The ideal $\{X \subseteq \omega_1 : |X| \leq \aleph_0\}$ is not precipitous.*

We shall consider a class of cardinal functions called *nice functions*. The following functions are nice: $\Phi(\alpha) = \omega_\alpha$; $\Phi(\alpha) =$ the αth weakly inaccessible cardinal. If Φ and ψ are nice, then so are, for example, $\psi_1(\alpha) =$ the αth fixed point of Φ; $\psi_2(\alpha) = \Phi(\alpha + \alpha)$; $\psi_3(\alpha) = \Phi(\psi(\alpha))$.

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THEOREM 2. *Suppose that there is a precipitous ideal I in $P(\omega_1)$. Let Φ be a nice cardinal function and let κ be a strong limit cardinal such that* $\mathrm{cf}(\kappa) = \omega_1$.

(a) *If $\kappa = \Phi(\omega_1)$, then $2^\kappa < \Phi((2^{\aleph_1})^+)$.*

(b) *If $\sup\{\Phi(\alpha): \Phi(\alpha) < \kappa\} < \kappa$, then $2^\kappa \leqslant \min\{\Phi(\alpha): \Phi(\alpha) > \kappa\}$.*

Suppose that I is also λ-saturated. Then

(c) *If $\kappa = \Phi(\omega_1)$, then $2^\kappa < \Phi(\lambda)$.*

(d) *If $2^{\aleph_0} \leqslant \Phi(\omega_1)$ and $\lambda < \Phi(\lambda)$, then $2^{\aleph_1} \leqslant \Phi(\lambda)$.*

(e) *If $2^{\aleph_0} \leqslant \Phi(\omega_1)$ and $\lambda = \Phi(\lambda)$, then $2^{\aleph_1} \leqslant \Phi(\lambda + \lambda)$.*

(f) *If $2^{\aleph_0} < \Phi(\alpha)$, where $\alpha < \omega_1$, then $2^{\aleph_1}$ is less than the αth value of Φ above λ.*

For instance, if Φ is the enumeration of fixed points of the aleph-function, (a) and (c) give estimates for a case left open in [2].

The assumption of niceness of Φ is necessary. Using the forcing constructions of [5], [6] and of Silver we can show: If κ is a super-compact cardinal and $\nu > \kappa$ arbitrary, then there exists a generic extension which preserves cardinals, and in which κ is a strong limit cardinal of cofinality ω_1, and $2^\kappa > \nu$.

We have two methods to prove our results. The first method uses a generic ultrapower—a combination of forcing and ultrapowers. This method is related to Silver's work on the singular cardinal problem [7] and Solovay's and Kunen's work on saturated ideals [8], [4].

We work in a given model M. Let I be a σ-additive ideal over ω_1 in M. We consider a generic extension corresponding to the partial ordering $\langle P(\omega_1)/I, \subseteq \rangle$. A generic set G is an ultrafilter (w.r.t. M). In $M[G]$ we consider the ultrapower $N = \mathrm{Ult}(M, G)$ and obtain an elementary embedding $i: M \longrightarrow N$. We use the lemma that N is well founded iff I is precipitous.

The second method is elementary in the spirit of [2] and uses almost disjoint transversals (a.d.t). Let I be σ-additive. An *I-function* is an ordinal function f such that $\mathrm{dom}(f) \in P(\omega_1) - I$. We set $f < g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(\alpha) < g(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$.

An I-function *f has κ almost disjoint transversals* if there exists a family F of I-functions such that: $|F| \geqslant \kappa$; $g < f$ for all $g \in F$; and if $g, h \in F$ and $g \neq h$, then $\{\alpha: g(\alpha) = h(\alpha)\} \in I$.

We make repeated use of the following lemmas.

LEMMA 1. *Suppose that f has κ a.d.t. and I is λ-saturated.*

(a) *If $\kappa \geqslant \lambda$, then for every $\nu < \kappa$ there exists $g < f$ which has ν a.d.t.*

(b) *If $\mathrm{cf}(\kappa) \geqslant \lambda$, then there exists $g \subseteq f$ such that for all $h \subseteq g$, h has κ a.d.t.*

LEMMA 2. *If I is precipitous and F is a nonempty family of I-functions* *closed under restrictions then there exists $f \in F$ such that there is no $g \in F$ with $g < f$.*

REMARK. After we had announced the above results, W. Mitchell proved that if it is consistent that a measurable cardinal exists, then it is consistent that ω_1 carries a precipitous ideal. In view of this result and of the Proposition above, it appears that the existence of a precipitous ideal over ω_1 is the exact counterpart of measurability, suitable for ω_1. This seems to give added interest to the results presented above.

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