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ON MAXIMAL FINITE IRREDUCIBLE SUBGROUPS OF GL(n, Z) I. THE FIVE AND SEVEN DIMENSIONAL CASE II. THE SIX DIMENSIONAL CASE

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By the Jordan-Zassenhaus Theorem there is only a finite number of conjugate classes (called **Z**-classes) of finite subgroups of GL(n, **Z**). After various authors have determined all of these groups for $n \leqslant 4$ [4], [3], as well as the maximal finite subgroups of GL(5, **Z**) [2], [7], [8], we develop new methods for the determination of the absolutely irreducible maximal finite subgroups of GL(n, **Z**) and compute these groups for $n = 5, 6, 7$. (We remark that irreducibility is tantamount to absolute irreducibility in case n is an odd prime number.) The algorithm proceeds in three steps.

1. Every absolutely irreducible finite subgroup G of GL(n, **Z**) fixes, up to scalar multiples, exactly one positive definite symmetric matrix $X \in \mathbf{Z}^{n \times n}$ called the form of G:

$$g^T X g = X \quad \text{for all } g \in G.$$

It follows that each maximal finite absolutely irreducible subgroup H of GL(n, **Z**) is the full **Z**-automorph of its form. (The **Z**-automorph of a positive form is certainly finite.) But the form of H is already determined by each of the absolutely irreducible subgroups of H. So at step 1 we determine all finite minimal absolutely irreducible subgroups of GL(n, **Z**) up to conjugacy under GL(n, **Q**), i.e. those absolutely irreducible groups which do not contain any proper absolutely irreducible subgroups. This is essentially a task of classical representation theory. As for the primitive groups we refer to papers by Brauer [1], Wales [9], and Lindsey [5]. To find the imprimitive groups we first had to prove an integral version of Clifford's Theorem. For $n = 5$ and 7 there are 2 minimal absolutely irreducible groups to be considered, but 33 for $n = 6$ because 6 is no prime so that many imprimitive groups turn up.

2. Step 2 consists of finding the **Z**-classes of the groups determined at step 1 which was done by means of electronic computation using the centering algorithm developed in [6]. Let us describe the algorithm in module theoretic terms. Let L and M be **Q**-equivalent **Z**G-representation modules, i.e. $\mathbf{Q}L \cong_{\mathbf{Q}G} \mathbf{Q}M$, then M is **Z**-equivalent to a submodule M' of L of finite index in L. One can

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choose M' in such a way that the prime divisors of $L : M'$ also divide $|G|$. Only such M''s are considered. If L is absolutely irreducible, a set of representatives of the **Z**-classes lying in the same **Q**-class as L is obtained as the set $R(L)$ of all those **Z**G-submodules M of L which are not contained in pL for any p dividing $|G|$. The computation of $R(L)$ requires the knowledge of the $\mathbf{Z}_pG$-composition factors of L/pL for all prime divisors p of $|G|$, say $A_1, \ldots, A_k$. Let $M \in R(L)$ and let $L = M_1 > M_2 > \cdots > M_s = M$ be a **Z**G composition series of L/M. Then the factor modules M_i/M_{i+1} ($i = 1, 2, \ldots, s - 1$) are isomorphic to certain A_j's ($j = 1, \ldots, k$). Hence M_{i+1} is obtained from M_i as the kernel of a **Z**G-epimorphism $\varphi_i: M_i \longrightarrow A_{j_i}$. Thus M_{i+1} can be obtained from M_i by solving systems of linear equations over a finite field. Each time a new M_{i+1} is obtained, one only has to test whether $M_{i+1} \in R(L)$(and need not compare M_{i+1} with any earlier M_k).

3. Having determined the **Z**-classes of the finite minimal irreducible subgroups one has to find the full **Z**-automorphs of their forms. They are the maximal finite irreducible subgroups of GL(n, **Z**).

For $n = 5$ the maximal finite absolutely irreducible subgroups of GL(n, **Z**) fall into 7 **Z**-classes forming 2 **Q**-classes of isomorphism types $C_2 \times S_6$ or $C_2 \sim S_5$. For $n = 7$ there are 7 **Z**-classes forming 3 **Q**-classes. The isomorphism types are $C_2 \times S_8$, $C_2 \sim S_7$ and the Weyl group $W(E_7)$. For $n = 6$ there are 17 **Z**-classes forming 9 **Q**-classes. The isomorphism types are $C_2 \sim S_6$, $(C_2 \times S_4) \sim C_2$, a subgroup of index 2 of $C_2 \sim S_6$, $(C_2 \times S_3) \sim S_3$, $C_2 \times W(E_6)$, $S_3 \times S_4 \times C_2$, $C_2 \times S_7$, $C_2 \times$ PGL(2, 7), $C_2 \times S_5$.

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