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Monotonie: Lösbarkeit und Numerik bei Operatorgleichungen, by Erich Bohl, Springer Tracts in Natural Philosophy, vol. 25, Springer-Verlag, Berlin, Heidelberg, New York, 1974, ix + 255 pp., $26.20.

Iterative methods for the solution of a linear operator equation in Hilbert space–A survey, by Walter Mead Patterson, 3rd, Lecture Notes in Mathematics, vol. 394, Springer-Verlag, Berlin, Heidelberg, New York, 1974, 183 pp., $8.20.

The books by Bohl and Patterson are detailed expositions of iterative methods for the solution of operator equations. The common thread between them is tenuous; the two books are surprisingly dissimilar in the point of view, scope and intent. The "Bibliography" in Patterson (which is intended to be complete with respect to the current literature for the topics that are covered) contains 145 references, while the "Literaturverzeichnis" in Bohl has 165 references. The intersection? Four standard textbooks in functional analysis (primarily for background material) and one paper: Kantorovič [**13**]. This 1939 paper (which was one of the first to realize the power of functional analysis methods in developing and unifying the theory of iterative methods) is certainly the beginning of the thin common thread in the books under review. In the spirit of the reviews in this Bulletin, a multiple review is expected to be a review of the "span" of the two books (rather than the union of two almost disjoint reviews). To put this "span" in a proper perspective, we first examine briefly some historic episodes and contemporary aspects of the broader area to which the two books belong. This is the field of numerical analysis in abstract spaces (alas, numerical functional analysis, abstract numerical analysis, or other names).

Numerical analysis and the traditional methods of computation have undergone structural changes over the past two decades. They are continuously being influenced by two definite trends, both of which received impetus, if not originated, about 1947–1948. The first is represented by the applications of abstract methods to some areas of numerical analysis. The second comes from computers and their logical structures. In this review, we shall be concerned only with the first trend.

In 1948, L. V. Kantorovič [**14**] published a fundamental long paper entitled *Functional analysis and applied mathematics*. This paper marks the beginning–and remains a landmark–in abstract numerical analysis. The thrust of the paper is to show explicitly that "the ideas and methods of functional analysis may be used for the construction and analysis of effective practical algorithms for the solution of mathematical problems with just as much success as has attended their use for the theoretical investigation (i.e., existence, uniqueness, etc.) of these problems. Moreover, results and error estimates from such a general point of view may in certain cases prove to be more complete and exact than those obtained for the separate special cases".

The beginning of modern numerical analysis took place about 1947. This was the date of the celebrated von Neumann-Goldstine paper [**25**]. About this time, Turing published a companion paper on roundoff error. (Incidentally, but significantly for the purpose of this review, this was also the date of the Kantorovič paper [**14**].) Some go back a bit more, and consider the zero date to be about 1936, which marks the appearance of Comrie's paper on the use of business machines for scientific computing and of Turing's first papers. About this time also, Ostrowski studied the numerical stability of Newton's method. (Again, incidentally, but significantly for abstract numerical analysis, this was about the time that Kolmogorov and others formulated the notion of a topological vector space, and Kantorovič published his results on iterations in functional analysis setting.)

Much of the foundations for work in abstract numerical analysis was laid in the early development of nonlinear functional analysis: the theory of differential and integral calculus in normed spaces, variational theory of nonlinear operators, theory of extrema of functionals, fixed point theorems, degree theory, etc. Historically, the beginnings of the theory of differentiation in infinite dimensional spaces go back to Volterra (1887), who introduced the concept of the variational derivative, and to Hadamard (1906), (1923), Fréchet (1911), (1912), (1925), Gâteaux (1913), (1919), Lévy (1911) and others. (The interested reader seeking detailed references may consult [**22**].) Graves (1927) extended Taylor's theorem to normed spaces; Hildebrandt and Graves (1927) generalized the implicit function theory; Kerner (1933) studied integrability conditions for abstract vector fields and generalized "Stokes' theorem" to Hilbert space, thereby introducing the concept of a gradient mapping or a potential operator. Golomb (1934) formulated the notion of gradient in function spaces, and Goldstine (1938), (1942) gave a "multiplier" rule in abstract spaces. Rothe (1948), (1953) studied topological properties of gradient mappings and the theory of extrema in Banach spaces. Topological methods in nonlinear analysis were also developed by Birkhoff and Kellogg, Leray and Schauder, Nemytskiĭ, Tikhonov, Krasnosel'skiĭ and others (see [**16**]). Variational methods for nonlinear equations were developed in the work of Hammerstein, Golomb, Lichenstein, Liusternik, Rothe, Sobolev and others (see [**34**]). The 1920's and early 1930's witnessed considerable interest in nonlinear problems in the setting of function spaces as indicated by the appearance of the celebrated paper by Birkhoff and Kellogg (1922) on invariant points in function spaces, the work of Graves and Hildebrandt, the development of differential calculus and the Leray-Schauder theory. Indeed, it may be said that the interest in that period was in nonlinear rather than linear functional analysis.

While there were no significant landmarks in the area of numerical functional analysis prior to the work of Kantorovič, the field was not completely without preestablished paths.

On the one hand, iterative and direct methods (for example, projection methods and best approximate solutions out of finite or simpler infinite dimensional subspaces, finite difference approximations, etc.) are often used in classical and functional analysis to establish existence theorems. When such methods are employed, one obtains contributions to existence theory and numerical procedures for approximate solutions of the problems. Striking examples of this interplay abound in the literature. We mention a celebrated paper by Courant, Friedrichs and Lewy (1941) for nonlinear hyperbolic equations and the early work of Rothe (1942) on parabolic equations in which existence theorems are established by showing that the solution of an approximating system of finite-difference and difference-differential equations converges uniformly to the solution of a partial differential equation.

The approximation techniques in these cited (and other related) contributions are less important for their own sake, than as a way to prove existence results. They were only pursued in the context of concrete problems. In addition, error analysis and rates of convergence, which are central topics in numerical analysis, were not considered. As such, these contributions are only precursors to numerical analysis in abstract spaces, although they represent significant advances in their own areas.

On the other hand, when existence theorems are established by (noniterative or nonconstructive) analytic or topological/variational methods, the setting itself (with a little strengthening of the hypotheses) is often capable of supporting useful iterative algorithms or constructive approximation schemes. This is another arena of numerical functional analysis. For example, the traditional method in calculus of variations for proving existence of a minimum of a functional is to establish precompactness of minimizing sequences and lower semicontinuity of the functional. This method carries over to more general settings (thanks to the weak-star topology, and to theorems of Alaoglu, Kakutani and Mazur). For example, let f be a weakly lower semicontinuous real valued function on a bounded weakly closed subset Ω (or in particular on a bounded, closed, and convex subset Ω) of a reflexive Banach space. Then f attains its infimum on Ω.

One aspect of numerical analysis in abstract spaces addresses problems of construction and approximation of the minimizing element of this minimization problem, and variants thereof, after existence has been settled. For some perspectives, see **[9]**, **[10]**, **[22]**.

The theory of nonlinear integral equations of Hammerstein type

$$(*) \qquad u(s) + \int_\Omega K(s, t) f(t, u(t))\, dt = y(s), \quad s \in \Omega,$$

has been since its inception one of the important areas of applications of the ideas and techniques of nonlinear functional analysis as well as of abstract numerical analysis. Hammerstein and others applied to $(*)$ the direct method of the calculus of variations, and various related results were obtained by the Russian school, particularly by Vaĭnberg and Krasnosel'skiĭ **[34]**, **[16]**. The integral equation $(*)$ has often been a testing ground for various iterative and approximation methods in nonlinear functional analysis, and occurs on several occasions in Bohl's book for that purpose.

The development of the topological degree, in the form of the homotopy invariance of the topological index, and the Leray-Schauder fixed point theorem for compact nonlinear mappings in Banach spaces have strongly influenced existence proofs in nonlinear analysis, since their formulation by Schauder and Leray in 1934. This theory has posed many challenging problems in abstract numerical analysis, and the constructive aspects still remain open except for a few special cases.

There were only a few other developments in nonlinear functional analysis (and the field remained almost dormant for nearly twenty years) until new and substantial advances were made during the fifties, sixties and early seventies in fixed point theory, monotone operators and convex analysis. These latter developments, along with an upsurge and new breakthroughs in topological degree theory, condensing and A-proper mappings, etc. dominate the contemporary scene of nonlinear functional analysis, and, as a corollary, some aspects of abstract numerical analysis. For perspectives and expository accounts of various contributions to nonlinear functional analysis, see **[6]**, **[30]**, **[38]**, [5], [32], **[28]**, [22] and **[8]**.

After the appearance of Banach's classic book, functional analysis was cultivated in a wide range of diverse directions. Soviet mathematicians, who have a strong tradition in this subject that goes back to its founding (witness Kreĭn-Milman, Naĭmark, Kolmogorov, Gel'fand, Tikhonov and others), pursued extensively applications to other fields of mathematics (as in the work of Sobolev, Kantorovič, Tikhonov, Krasnosel'skiĭ, Vaĭnberg and others); see **[20]**. By contrast American and Western mathematicians in that period were primarily concerned with the structure of functional analysis, the theory of topological vector spaces, and the achievement of greater generality and abstractness. We did not have counterparts to Sobolev, *Applications of functional analysis to mathematical physics*, or to the books by Kantorovič and Akilov **[15]**, Krasnosel'skiĭ **[16]**, and Vaĭnberg **[34]**. The Soviets, by contrast, did not have counterparts to Kelley, Namioka, et al., nor to Bourbaki. Kantorovič and Akilov write in the Foreword to **[15]**, " . . . the theory of linear topological spaces, discussed in Chapter XI, is the only one that strikes an echo in the present book. There is virtually no literature in Russian on this subject . . . ". (Even this brief treatment is in the spirit of normed spaces, and the Russian graduate student seriously interested in linear topological spaces had to rely on expositions by Western authors.) By contrast, many American graduate students in the fifties and early sixties studied normed linear spaces from Liusternik and Sobolev, Kolmogorov and Fomin, or Kantorovič and Akilov. Such things as Fréchet differentials, implicit function theorem (of Hildebrandt and Graves), etc., did not find their way to American textbooks in functional analysis of that era. One should mention, however, the various New York University lecture notes by Friedrichs on applications of functional analysis to quantum field theory, differential equations, and other areas of analysis.

Considerable Soviet efforts have been directed to numerical functional analysis since the 1940's (see **[20]**). Applications of functional analysis to numerical mathematics were also extensively cultivated by Collatz and the German school (cf. [7]). Bohl is one of its graduates.

In the United States, there was little interest in numerical functional analysis during the early development. (I recall two seminars that I attended as a graduate student in 1961. The TVS seminar was fully packed; a seminar to read Kantorovič paper, etc., had two students.) The field was viewed with a mixture of suspicion and indifference. Functional analysts dismissed it as "too soft"; numerical analysts claimed it was "not useful". Papers in the field used to be reviewed in the Math. Reviews under "Miscellaneous applications of functional analysis".

This situation has changed drastically over the past fifteen years, which have witnessed considerable interest in, and substantial contributions to, numerical analysis in abstract spaces. The field has become sufficiently popular to receive its separate classification number (65J) and to sport its own column in the Math. Reviews (on equal standing with such things as Abelian categories or Tauberian theorems). The relevance of this field to numerical analysis has been acknowledged by leading numerical analysts (A. S. Householder, in his review of [7] in MR, writes "It seems strange that this book should be the first of its kind, since it hardly needs to be said that numerical mathematics must draw heavily from functional analysis."). The field has also gained membership in the functional analysis club by inducing developments in functional analysis whose origin resides in questions of approximation-solvability (see, e.g., Petryshyn [**28**]). The field now has depth and breadth of its own.

This change was brought about by at least three developments: the rising demand for approximate and iterative solutions of operator equations; the impact of functional analysis on areas of approximation, optimization and control theory, integral equations, etc.; and the substantial recent advances, alluded to earlier, in nonlinear functional analysis.

Over the past few years, developments in numerical functional analysis have been going side by side with developments in these fields, and there is now much greater interplay among these disciplines. In "numerical analysis and approximation theory in abstract spaces" there is now such intensive activity on many fronts that a closer overview of the whole battlefield would show only smoke. (For various perspectives, the interested reader may refer to [**1**], [**2**], [**3**], [**4**], [**7**], [**9**]–[**12**], [**18**], [**19**], [**24**], [**26**]–[**31**], [**33**], [**37**].)

The books by Bohl and Patterson are expository accounts of some advances made during the past ten years on two different fronts of numerical functional analysis.

Bohl considers monotone operators on ordered structures, and related topics dealing with solvability and numerical methods, primarily for nonlinear operator equations. The setting throughout involves a combination of concepts of distances and of orderings. The first third of the book (Chapters 0–III) gives a review of prerequisites and standard results. Chapters IV and V deal with iterative methods for P-bounded operators and for monotone-decomposable operators. We describe next some results in this direction.

Let $(X, \|\cdot\|)$ be a real normed space and let K be a closed cone in X. A partial ordering is induced on X by $x \leqslant y$ iff $y - x \in K$. For $e \geqslant 0$ and $e \neq 0$, denote by X_e the subset of elements $x \in X$ for which there is a real number p such that $\pm x + pe \geqslant 0$. X_e is a normed space with the norm $\|x\|_e$ defined as the infimum of the p's satisfying the above inequality. A cone K is called normal, if there exists a positive number α such that $x, y \in X$ and $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \alpha\|y\|$.

THEOREM. *Let $(X, \|\cdot\|)$ be a normed space which is ordered by a closed normal cone K. Let $T : D \to D$ be a P-bounded mapping of a closed subset D of X into itself (i.e., it satisfies the following condition: There exists a syntone linear operator $P : X \to X$, in the sense that $v \leqslant w$, $v, w \in X \Rightarrow Tv \leqslant Tw$, such that for every $e \geqslant 0$, $e \neq 0$ and all $x, y \in D$, $x - y \in X_e$,*

$$Tx - Ty \leqslant \|x - y\|_e Pe.$$

Let $x_0 \in D$ and suppose that there exist a $z \geqslant 0$ with $(I - P)z \geqslant 0$, $e \neq 0$ and $x_0 - Tx_0 \in X_{(I-P)z}$. Suppose that one of the following conditions holds:

(i) *T and P are completely continuous;*

(ii) *T is completely continuous and* $\lim_{n\to\infty}(I - P)P^n z = 0$;

(iii) *X is complete and* $\lim_{n\to\infty} P^n z = 0$.

Then T has a fixed point $x^* \in D$. Moreover, the sequence $x_n = T^n x_0$ converges to x^* under hypothesis (iii), *while a subsequence converges to x^* under hypothesis* (i) *or* (ii). *The following error estimate holds for all cases:*

$$\|x_n - x^*\|_{P^n z} \leqslant \|x_0 - x_1\|_{(I-P)z}.$$

The result corresponding to condition (iii) is a variant of the classical contraction theorem, condition (i) gives a variant of the Schauder theorem, while condition (ii) yields an intermediate theorem.

The powerful theory of monotone decomposable operators on ordered spaces yields two-sided estimates for solutions. Let T be an operator on a partially ordered Banach space X and suppose that T has the form $T = T_1 + T_2$, where T_1 is syntone and T_2 is antitone ($v \leqslant w \Rightarrow Tv \geqslant Tw$), and T_1 and T_2 are continuous on a convex subset Ω of X. Consider the iterations $v_{n+1} = T_1 v_n + T_2 w_n$ and $w_{n+1} = T_1 w_n + T_2 v_n$, where v_0, w_0 are initial approximations in Ω with $v_0 \leqslant v_1 \leqslant w_1 \leqslant w_0$. If T maps the interval $M_n = [v_n, w_n]$ for some $n \geqslant 0$ into a relatively compact set, then T has at least one fixed point in M_n.

Chapter 6 (78 pp.) deals with iterative solutions of systems of nonlinear equations on R^n including such topics as the spectral radius of a nonnegative matrix, discretization of boundary value problems and practical execution of error estimates. For a nonlinear mapping $F : D \to D (D \subset R^n)$, several sufficient conditions for the existence of a fixed point in D are given. Componentwise error estimates for the classical method of successive approximations, the Gauss-Seidel and related iterations use algorithms involving the matrix of Lipschitz constants for F, i.e., a matrix P for which $\|F(s) - F(t)\| \leqslant P\|s - t\|$ componentwise ($s, t \in D$). The proofs use the partial order structure of R^n, thus achieving more generality than the usual norm structure provides. Several numerical examples are presented, together with applications to linear systems of equations and to differential equations. Some of the results of the earlier chapters are specialized and sharpened for R^n.

Chapter 7 deals with applications to linear and nonlinear integral equations, again in the framework of normed ordered structures.

Patterson conducts a survey of iterative techniques for solving the operator equation $Ax = y$ in the context of real or complex Hilbert spaces, where A is a linear (and bounded or densely-defined unbounded) operator. No order properties on the spaces are assumed. The primary concern is with the convergence of the sequence of approximate solutions, and in some cases also with estimates of the error and the speed of convergence. The methods are all theoretical in nature. No attempt is made to discuss practical applications of the techniques considered, or to examine numerical or computational characteristics of these methods (in contrast with some parts of Bohl's book that are devoted to such aspects). Patterson provides an overview of the wide range of techniques in solving linear operator equations by iteration, and brings together in an organized collection the numerous but widely scattered results now available in the field. Methods of successive approximations and gradient methods form a central core in this collection while nearly one third of the book is devoted to papers of Petryshyn, which unify and extend several iterative methods and introduce others.

The earliest convergence theorem for the method of successive approximations in the context of linear operator equations is the following result of Picard-Poincaré-Neumann. Let T be a linear operator on a Banach space X such that $\|T\| < 1$. Then the equation $x - Tx = y$ has a unique solution for every $y \in X$, given by $x^* = \Sigma_{n=0}^{\infty} T^n y$. That is, the sequence $\{x_n\}$ given by $x_{n+1} = Tx_n + y$, x_0 arbitrary, converges to x^*.

This result has been weakened and generalized in many directions. Krasnosel'skiĭ (1960) was able to use the weaker hypothesis $\|T\| \leqslant 1$, but with added restriction that T is a selfadjoint operator in Hilbert space such that -1 is not an eigenvalue of T, to prove that the sequence $\{x_n\}$ converges to a solution (not necessarily unique) for each y in the range of $I - T$. Browder and Petryshyn (1966) proved a much stronger theorem in Banach space which subsumes this result. Let T be a bounded linear operator on a Banach space X such that $\{T^n x\}$ converges strongly for all $x \in X$. If y is in the range of $I - T$, then the sequence $\{x_n\}$ converges to a solution of $(I - T)x = y$. If any subsequence of $\{x_n\}$ converges, the whole sequence converges. If X is reflexive and $\{x_n\}$ is bounded, then $\{x_n\}$ always converges to a solution.

Another direction for generalizations involves replacing the classical successive approximation by an averaging iterative procedure via a Toeplitz matrix. Let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_0 = 1$, $0 < \alpha_n \leqslant 1$, and $\Sigma_{n=0}^{\infty}\alpha_n$ diverges. The iteration scheme is of the form $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n)x_n$, x_0 arbitrary. Equivalently, $x_{n+1} = \Sigma_{k=0}^{n} t_{nk} T x_k$, where $t_{nk} = \alpha_k \Pi_{i=k+1}^{n}(1 - \alpha_i)$, $k < n$, α_n for $k = n$, and 0 for $k > n$. (The matrix (t_{nk}) is a Toeplitz matrix.) Of particular interest is the averaging iteration, first considered by Mann (1953), $x_{n+1} = (n + 1)^{-1}\Sigma_{k=0}^{n} T x_k$, in which the special Toeplitz matrix is a Cesàro matrix. Several generalizations of the fundamental result of Browder and Petryshyn have also been given. For the most part, these generalizations are based on an averaging technique of Mann and use various ergodic theorems (G. Birkhoff, Eberlein, Yosida-Kakutani) or variants thereof. Patterson gives an exposition of a generalization, due to de Figueiredo and Karlovitz (1968), which is based on the Yosida-Kakutani Mean Ergodic Theorem.

The last third of Patterson's book is devoted to gradient methods. This class of iterative methods includes the method of steepest descent and the method of conjugate gradients which is a special case of the conjugate direction method.

Let H be a complex Hilbert space, and let $A : H \to H$ be a bounded, positive bounded below linear operator. Then the equation $Ax = y$ has a unique solution x^* for every $y \in H$. Let $E(x) = \langle e, r\rangle = \langle e, Ae\rangle$, where $e = x^* - x$. In the gradient methods we minimize $E(x)$ successively in the direction of linearly independent vectors $p_0, p_1, \ldots$ which are to be determined. Thus we consider an iterative scheme of the form

$$x_{n+1} = x_n + \alpha_n p_n, \qquad \alpha_n = \langle r_n, p_n\rangle / \langle p_n, Ap_n\rangle.$$

In the method of steepest descent, we select $p_n = r_n$ since this is the direction of maximum change for $E(x)$ at x_n. In the conjugate direction method, we select p_n's so that $\langle p_m, Ap_n\rangle = 0$ for $n \neq m$. In the conjugate gradient method, the p_n's are chosen iteratively:

$$p_{n+1} = r_{n+1} + b_n p_n, \qquad b_n = -\langle r_n, Ap_n\rangle / \langle p_n, Ap_n\rangle.$$

Various convergence proofs obtained by several authors are presented, along with geometric rates for the steepest descent and conjugate gradient methods. The presentation is laced with historical comments on these methods and their variations. Generalizations to certain classes of densely defined unbounded operators are also given by using modifications of the renorming techniques of Friedrichs. The author cites recent extensions of these methods to least squares solutions of linear operator equations and relations to generalized inverses (cf. [**28**], [**29**] and references cited therein) but does not discuss these matters since he confines his consideration to equations which are uniquely solvable.

Patterson's book is based on his expository dissertation at Syracuse University; it provides a strong argument for merits of such dissertation. It presents an excellent and balanced overview of one on the outside looking in. But there is also more: simplifications of some proofs, unification of several approaches, and a leisurely pace which makes for enjoyable reading. Bohl's book, on the other hand, is based to a great extent on the research of the author and the German school, with peripheral remarks on related works. (The English spelling of Diskrete, the modesty in not including Bohl in the Authors Index, and other colorful matters in Bohl's book provide a counterexample to a proof in a recent review in this Bulletin that Springer-Verlag has finally become a naturalized U. S. publisher. On the other hand, perhaps there is some truism in Thomas Wolfe's novel, "You can't go back home again".)

Both books may serve as textbooks for graduate courses or seminars and as useful references in constructive methods for operator equations. These methods (and numerical functional analysis in general) are still based on the continuum and the nonconstructive theory of functional analysis. All this should be distinguished from constructivism in mathematics (cf. [**3**]) that has not yet directly influenced numerical functional analysis; perhaps the possibilities have not been explored.

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Comparison theorems in Riemannian geometry, by Jeff Cheeger and David G. Ebin, North-Holland Mathematical Library, vol. 9, North-Holland, Amsterdam; American Elsevier, New York, 1975, viii + 174 pp., $20.95.

Differential geometry is an almost unique area within mathematics, since it involves both the old and the new in an essential way. Riemannian geometry itself has, of course, been around for over one hundred years: About twenty-five years ago geometers began to ask how the local curvature of Riemannian manifolds could influence their global properties. (There were clues that this was an interesting question, e.g., the theorem of Hadamard and Cartan that a complete simply connected Riemannian manifold of nonpositive sectional curvature was diffeomorphic to Euclidean space.) The major opening salvo in this campaign was Rauch's work, published in 1951, showing that a (positive definite) Riemannian manifold whose sectional curvature function is sufficiently close to the curvature of the usual metric on the sphere is, in fact, homeomorphic to the sphere. Rauch combined techniques whose roots lie in the classical work: Sturm-type theorems for the systems of linear ordinary differential equations which result from linearization of the geodesic equations, and the distance minimizing property of the geodesics. Berger, Klingenberg and Toponogov then developed the conditions on the curvature which assure that the manifold is homeomorphic to the sphere and analyzed what happens at the precise point that the conditions are violated. They also developed a refined and powerful methodology to deal with this type of problem. In the sixties the methods were successfully applied to two general problems: Find Rauch-type conditions on the curvature which would assure that the manifold is diffeomorphic to the sphere, and study general global