BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 5, September 1978

FIBRATIONS AND GEOMETRIC REALIZATIONS

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There is a folk theorem associated to the construction of classifying spaces for topological groups which says that a map of simplicial spaces which is a fibration in every degree has a fibration as its geometric realization. Peter May [GILS] has given a useful form of this which involves quasifibrations. This result has seemed to me a very interesting one, as it relates two rather opposite types of operations. On the one hand, it involves the geometric realization, which is defined by mapping into other things, and on the other hand, it involves fibrations, which are characterized by the properties of maps of other things into them. Any theorem which mixes "left" and "right" mapping properties should be expected to be difficult to prove. Since what is really wanted in applications is a homotopy theoretic result, the possibilities for complication are almost infinite.

It seemed to me that a proof should be found which allowed for a homotopy invariant statement of the theorem, and which had the property that the ad hoc part of the argument was isolated in a reasonably small, isolated computation. After some effort, I managed to find such a proof, which I will outline in this paper. The virtue of this proof is that it involves a number of areas of topology which are known only to experts and which deserve a wider audience. Thus the proof of the theorem about the geometric realization of fibration has become the occasion for an exposition of simplicial methods, axiomatic homotopy theory, and homotopy limits and colimits. I shall only touch on simplicial methods, as there are several texts which cover these, and spend most of my time on axiomatic homotopy theory and the theory of homotopy limits and colimits. These are subjects which I think will become more pervasive in topology as attempts are made to apply homotopy theory in ever more general settings.

For those who know what simplicial objects are, I will state the result on geometric realizations and illustrate it with some examples. For those not familiar with the terms I shall attempt to explain them as the talk proceeds.

THEOREM. *If f: $X \to Y$ is a map of simplicial spaces such that $\pi_0(f)$ is a Kan fibration, and if the higher groupoids $\Pi_\infty(X)$ and $\Pi_\infty(Y)$ are fully fibrant, then for any map g: $Y' \to Y$ of simplicial spaces, if X' is the homotopy theoretic fiber product of Y' with X over Y, $R(X')$ is the homotopy theoretic fiber product of $R(Y')$ with $R(X)$ over $R(Y)$, where R denotes the geometric realization.*

This theorem is proved for bisimplicial sets to take advantage of the rather

An Invited Address delivered at the 81st Annual Meeting of the American Mathematical Society at Washington, D. C., January 25, 1975; received by the editors October 31, 1977.

AMS (MOS) subject classifications (1970). Primary 55-02; Secondary 55Dxx.

nice properties of the geometric realization functor. However, the singular complex and the standard realization functor carry it over to topological spaces with the homotopy type of CW-complexes, since the results and the hypothesis are both homotopy theoretic.

The groupoids Π_∞ are not familiar to many people. They are the wreath product of the fundamental groupoid with the product of the higher homotopy groups. The condition that a simplicial groupoid be fully fibrant is somewhat difficult to explain, but if a simplicial groupoid is equivalent to a simplicial group, this condition will always be satisfied. On the other hand, if the simplicial groupoid is equivalent to a discrete simplicial groupoid (one object and one morphism in each component), the condition that it be fully fibrant is vacuous. Thus the theorem applies whenever $\pi_0(f)$ is a Kan fibration and X and Y satisfy the condition that either each X_n be discrete or connected and each Y_n be either discrete or connected. Alternatively, we could ask that each X_n or Y_n be a topological group. One of the most useful cases, of course, is the one treated by May in [GILS], where X_n is a point for all n and Y_n is connected for all n.

As a special case of our theorem, consider the case when X, Y, and Y' are discrete spaces. Then $\pi_0(X)$ and $\pi_0(Y)$ are the underlying simplicial sets, and $R(X)$ and $R(Y)$ are their usual geometric realizations. Thus, as a special case, we obtain the theorem that the ordinary geometric realization from simplicial sets to topological spaces preserves homotopy theoretic fiber products. This is not actually a new result; it follows from a result of Quillen's which states that the geometric realization of a Kan fibration between simplicial set is a Serre fibration of topological spaces.

The reader is encouraged to prove the lemmas which I state here for abstract homotopy theory. I learned them from various people in one way or another, particularly Dan Quillen, Dan Kan, Pete Bousfield, Ken Brown and Chris Reedy. The proofs are all elementary, though a few of them (particularly the pasting lemma) do require drawing some rather complicated diagrams. The responsibility for the correctness of these results rests with me; if there are errors which have crept in, they are my own. The results on the existence of homotopy colimits and limits in abstract homotopy theory are also my own, and a more detailed account of these, together with some results on "functors up to homotopy" will appear elsewhere. There is, of course, considerable overlap with the work of Bousfield and Kan, whose work on homotopy limits and colimits represents a high point in homotopy theory.

1. Simplicial topology. It has long been known that many important topological spaces can be studied combinatorially by explicitly considering decompositions of these spaces into unions of Euclidean discs. There are many familiar spaces which are homeomorphic to the Euclidean disc; among these, the simplices are generally the easiest to use since any map of the vertices of a simplex into a linear space has a unique linear extension to the simplex. The study of spaces which have been decomposed into simplices has been extended to the study of abstract simplicial sets. Up to homeomorphism, the abstract simplicial sets are no more general than triangulated spaces, but they are more flexible with respect to the formation of quotient spaces, and they lend themselves better to combinatorial analysis than do the triangulated spaces.

By a geometric simplex of dimension n, we mean a space with a fixed homeomorphism with the set of $(n+1)$-tuples $(t_0, \ldots, t_n)$ of nonnegative real numbers such that $t_0 + \cdots + t_n = 1$. We write Δ^n for the standard simplex of dimension n which is just the space of such $(n+1)$-tuples. By a face of Δ^n we mean a subspace defined by $t_{i_1} = \cdots = t_{i_k} = 0$ for some (possibly empty) subset $(i_1, \ldots, i_k)$ of $(0, 1, \ldots, n)$. If the subset is not empty, the corresponding face is called a proper face as it is not equal to Δ^n. The union of all the proper faces is called the boundary of Δ^n, and it is homeomorphic to the sphere of dimension $n-1$.

A finite triangulation of a topological space X is a homeomorphism between X and a subspace of some Δ^n which is the union of a set of faces of Δ^n. Not every space has a finite triangulation; those spaces which do are called polyhedra. Included among the polyhedra are such spaces as the compact differential manifolds.

Associated to a triangulation of a topological space X is a combinatorial structure called a simplicial complex. A simplicial complex K is any sublattice of the lattice of finite subsets of a given set S which has the following property: $A \subset B \subset S$, $B \in K$ implies $A \in K$. If we regard X as a subspace of Δ^n which is the union of certain faces of Δ^n, we associate to this triangulation the set S of all faces of dimension 0 (called the vertices of Δ^n) which lie in X and let $B = (v_1, \ldots, v_r)$ be in K if and only if the r-dimensional face containing B lies in X.

Simplicial complexes may or may not be considered to have an ordering on their vertices. We shall consider all simplicial complexes to be equipped with orderings on their vertices, and all maps to preserve these orderings.

A simplicial complex K has associated to it a topological space $|K|$ called the geometric realization of K. In the example above, X is naturally homeomorphic to the geometric realization of its associated simplicial complex, so we see that no information is lost if we consider simplicial complexes rather than triangulated spaces.

In order to form the geometric realization of a simplicial complex K, take the disjoint union of one copy Δ^{n_σ} of a simplex of dimension n_σ for each $\sigma \in K$, where n_σ is the number of elements in σ minus one. We regard the elements of σ as the vertices of Δ^{n_σ}, and for $\sigma \subset \tau \in K$, we identify Δ^{n_σ} with the corresponding face of Δ^{n_τ}. The resulting quotient space is the geometric realization $|K|$ of K.

While not every map $|K'| \to |K|$ between two simplicial complexes is homotopic to a map induced from a map of vertices, every map can be represented as the geometric realization of a map of simplicial complexes if one performs the combinatorial construction known as subdivision often enough to the source simplicial complex. Thus, not only can the polyhedra be recovered from simplicial complexes, but the homotopy theory of polyhedra can be recovered from the homotopy theory of simplicial complexes. This shows that, at least for those polyhedra for which triangulations are known, homotopy theory can be reduced to combinatorics. Indeed, results of Ed Brown [FC] imply that the computations of homotopy groups of polyhedra are recursively computable by combinatorial methods.

Unfortunately, the category of simplicial complexes is not sufficiently flexible to allow for some of the standard constructions of the homotopy theorist. The main deficiency is the lack of finite colimits. For example, if K' is a subsimplicial complex of a simplicial complex K, there is no simplicial complex K/K' whose geometric realization will be $|K|/|K'|$. To remedy this, another combinatorial construction was introduced which is an extension of the idea of a simplicial complex.

For the cateogrically minded, simplicial sets may be regarded as the sheaves on the category of ordered simplicial complexes with respect to the canonical topology. For such people, the rest of this section will follow immediately from first principles, and they may skip to the sections on homotopy theory.

There are certain simplicial sets $P(\mathbf{n})$ which are the power sets of the sets $\mathbf{n} = \{0, 1, \ldots, n\}$. That is, $P(\mathbf{n})$ is the lattice of all subsets of $\mathbf{n}$. Then $|P(\mathbf{n})| = \Delta^n$. If K is any simplicial complex, let $K_n = \mathrm{Hom}(P(\mathbf{n}), K)$. Then inside K_n one has a subset consisting of all monomorphisms $P(\mathbf{n}) \to K$. Since we have ordered the vertices, these correspond to the n-dimensional simplices of $|K|$. Every element f: $P(\mathbf{n}) \to K$ of K_n can be described as $f = gP(\mathbf{h})$ where g: $P(\mathbf{m}) \to K$ is a monomorphism, and $m + 1$ is the number of vertices in the image of f. The elements of K_n which correspond to monomorphisms are referred to as the nondegenerate simplices of K in dimension n; the remainder are called the degenerate simplices in dimension n.

One can take the product of simplicial complexes. If K and L are two simplicial complexes, $K_n \times L_n = (K \times L)_n$ for all n. However, the nondegenerate simplices do not follow this rule. The standard map $|K \times L| \to |K| \times |L|$ is a bijection, and if either K or L is finite, it is a homeomorphism.

By a simplicial set X, we mean a contravariant functor from the category $\mathcal{O}$ of finite ordered sets to the category of sets. We write X_n for $X(\mathbf{n})$. Notice that since simplicial sets are functors into the cateogry of sets, they inherit from the category of sets all limits and colimits. The following properties of simplicial sets are quite straightforward to establish:

(1.1) (Yoneda Lemma). *If Δ^n is the simplicial set given by $\Delta^n_m = \mathrm{Hom}(\mathbf{m}, \mathbf{n})$, there is for all X a natural isomorphism $X_n \cong \mathrm{Hom}(\Delta^n, x)$.*

(1.2) *The functor $K \mapsto K_.$, where $K_.(\mathbf{n}) = K_n$ from the category of ordered simplicial complexes to the category of simplicial sets is full and faithful (i.e.,* $\mathrm{Hom}(K', K) = \mathrm{Hom}(K'_., K_.)$), *and it preserves limits.*

(1.3) *Every subsimplicial set of a simplicial complex is a simplicial complex.*

(1.4) *For any simplicial set X, there are simplicial complexes K', K and maps f, g: $K' \to K$ such that X is the coequalizer of $f_., g_.$: $K'_. \to K_.$.*

If X is a simplicial set, the elements of X_n are called the n-simplices of X. If there is a proper epimorphism h: $\mathbf{n} \to \mathbf{m}$ and an element $\tau \in X_m$ such that $\sigma = X(h)(\tau)$, the n-simplex σ of X is called degenerate. If no such pair (h, τ) exists, σ is called nondegenerate.

There is often confusion as to why the degenerate simplices are included in a simplicial set. The reason is fairly simple once one attempts to construct maps between simplicial sets. Notice that Δ^0 is the terminal simplicial set and plays the role of the one point space (indeed, $|\Delta^0| = \Delta^0$ is the one point space). If we look at the constant map $\Delta^n \to \Delta^0$, this is the geometric realization of the dimensionwise constant map $f\colon \Delta^n \to \Delta^0$. If we look at the behavior of f on simplices, Δ^n has in dimension m exactly $\binom{n+1}{n+1}$ nondegenerate simplices. Thus, for $0 \leqslant m \leqslant n$, $f_m\colon \Delta^n_m \to \Delta^0_m$ has to have nonempty image for the nondegenerate simplices. However, in positive dimensions, Δ^0 has no nondegenerate simplices so that in order to describe the constant map $f\colon \Delta^n \to \Delta^0$, we must include in Δ^0 the degenerate simplices.

We can extend the geometric realization functor from the category of simplicial complexes to the category of simplicial sets as follows. If X is a simplicial set, let $\mathrm{Cov}(X)$ be the disjoint union of one copy of Δ^{n_σ} for all simplices $\sigma \in X_{n_\sigma}$ of X, and let $p\colon \mathrm{Cov}(X) \to X$ send the identity element of Δ^{n_σ} to σ. Then $\mathrm{Cov}(x)$ is a disjoint union of finite simplicial complexes. Let $\mathrm{Rel}(x)$ be the fiber product over X of $\mathrm{Cov}(x)$ with itself, and let r, s: $\mathrm{Rel}(x) \to \mathrm{Cov}(x)$ be the two projections. Then $\mathrm{Rel}(x)$ is a disjoint union of finite simplicial complexes. Let $|X|$ be the quotient of $|\mathrm{Cov}(x)|$ by the relation $|r|$, $|s|$: $|\mathrm{Rel}(x)| \to |\mathrm{Cov}(x)|$ (i.e., $|X|$ is the coequalizer of $|r|$ and $|s|$). This can be shown to agree with the geometric realization functor defined previously on finite simplicial complexes. Further, this functor has a right adjoint given by the singular complex functor: $\mathrm{Sing}(A)_n = \mathrm{Hom}(\Delta^n, A)$. This geometric realization preserves colimits. It also can be shown to preserve finite limits in the full subcategory of CW-complexes.

In many ways, simplicial sets are much like CW-complexes. Indeed, every CW complex has the homotopy type of the geometric realization of its singular simplicial set. Further, one can define skeleta for simplicial sets as follows. If X is a simplicial set, the p-skeleton $\mathrm{sk}^p(X)$ is, in degree n, the set of n simplices of X which come from simplices of dimension q for $q \leqslant p$ by composition with a map $\Delta^n \to \Delta^p$. If $\partial\Delta^n = \mathrm{sk}^{n-1}(\Delta^n)$, $(|\Delta^n|, |\partial\Delta^n|)$ is homeomorphic to (B^n, S^{n-1}), where B^n is the n-ball, S^{n-1} is the $(n-1)$-sphere. Further, $\mathrm{sk}^n(x)$ is obtained from $\mathrm{sk}^{n-1}(x)$ by attaching along a map $a_\sigma\colon \partial\Delta^{n-1} \to \mathrm{sk}^{n-1}(x)$ one copy of Δ^n for each nondegenerate simplex σ of X of dimension n.

There is a straightforward homotopy theory for simplicial sets if one uses Δ^1 as the unit interval to define homotopies. There are technical difficulties, however, which make this approach rather unsatisfactory. For example, one cannot "turn over" Δ^1 to reverse the direction of homotopies, so that homotopy is not a symmetric relation. Also if one joins two copies of Δ^1 along one end, the result has the same geometric realization as Δ^1, but is not isomorphic to Δ^1. However, there is a subdivision functor for simplicial sets just as there is for simplicial complexes. The obvious thing to try works. If X, Y are simplicial sets with a finite number of nondegenerate simplices in X, then for every $f\colon |X| \to |Y|$ there is an n such that f is homotopic to the geometric realization of a map of the nth subdivision of X to Y. If we also look at subdivisions of $X \times \Delta^1$, we find that the resulting homotopy classes of maps of subdivisions of X into Y is the same as the homotopy classes of maps of $|X|$ into $|Y|$. Thus, we see that if we allow for sufficient subdivision, the homotopy theory of polyhedra can be described entirely in terms of the homotopy theory of simplicial sets, with the geometric realization and singular functors expressing this equivalence.

Following Kan, one can describe the homotopy theory of simplicial sets in a neater, if less intuitive manner. Kan points out that the subdivision functor has a right adjoint Ex, which he calls the extension functor. The projection of the subdivision to the identity gives a natural transformation $Y \to \mathrm{Ex}(Y)$ such that for all Y, $|Y| \to |\mathrm{Ex}(Y)|$ is a homotopy equivalence and a cellular inclusion of CW complexes. He then iterates Ex, and lets Ex^∞ be the colimit of the Ex^n for $n \geqslant 1$. Adjointness now shows us that for X any finite simplicial complex (or simplicial set with finitely many nondegenerate simplices), the homotopy classes of maps of X into $\mathrm{Ex}^\infty(Y)$ will for all Y be the same as the homotopy classes of maps of $|X|$ into $|Y|$. Thus we obtain a second description of the homotopy theory of polyhedra, though at a price–even if Y is a simplicial complex, $\mathrm{Ex}(Y)$ will in general not be a simplicial complex, but only a simplicial set. However, $\mathrm{Ex}^\infty(Y)$ can be defined, while no infinite subdivision functor can be defined.

There is another way to describe the homotopy theory of simplicial sets. Call a map f': $X \to Y$ a weak equivalence if either of the following equivalent conditions hold:

(1.5) $\mathrm{Ex}^\infty(f)$ is a homotopy equivalence.

(1.6) $|f|$ is a homotopy equivalence.

Then, as was pointed out by Gabriel and Zisman [CFHT], the homotopy category for simplicial sets can be defined as the localization of the category of simplicial sets with respect to the weak equivalences.

2. Abstract homotopy theory. It is possible to describe the simplest features of homotopy theory entirely in terms of which maps are to be considered to be "homotopy equivalences". For example, if f, g: $A \to B$ are two maps, we could say that f is homotopic to g if there is a space A' and two "homotopy equivalences" i, j: $A \to A'$, together with a map F: $A' \to B$, such that $Fi = f$, $Fj = g$. This introduces a reflexive, symmetric relationship on maps, though not necessarily a transitive one. Further, our "homotopy equivalences" need not have homotopy inverses. For this reason, we shall use the term "weak equivalence" rather than the term "homotopy equivalence", since the latter term carries the connotation of a map with some type of inverse.

We shall say that a category $\mathcal{C}$ has a notion of weak equivalence if certain of its morphisms have been called weak equivalences and if the following two axioms hold:

(WE 1) Every isomorphism is a weak equivalence.

(WE 2) If f, g are morphisms in $\mathcal{C}$ such that fg is defined, then if two of f, g, fg are weak equivalences, so is the third.

Associated to a cateogry $\mathcal{C}$ which has a notion of weak equivalences is another category $\mathrm{Ho}(\mathcal{C})$ which is obtained from $\mathcal{C}$ by localizing with respect to (i.e., inverting formally) the weak equivalences of $\mathcal{C}$. There is a localizing functor γ: $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ which takes all weak equivalences (and possibly other morphisms) to isomorphisms. If $\mathcal{C}$ is the category of simplicial sets and weak equivalences are taken to be those maps of simplicial sets whose geometric realizations are homotopy equivalences, $\mathrm{Ho}(\mathcal{C})$ is equivalent as a category to the category whose objects are the CW-complexes and whose maps are the homotopy classes of maps between spaces. If $\mathcal{C}$ is taken to be all topological spaces, and weak equivalences are taken to be homotopy equivalences, then $\mathrm{Ho}(\mathcal{C})$ is the category in which the maps are homotopy classes of maps and all spaces occur as objects in $\mathrm{Ho}(\mathcal{C})$. For the obvious reason, we refer to $\mathrm{Ho}(\mathcal{C})$ as the homotopy category of $\mathcal{C}$, and if f is a morphism in $\mathcal{C}$, we call $\gamma(f)$ the homotopy class of f.

Even at this level, it is possible to exhibit some of the features which separate homotopy theory from ordinary category theory. Suppose that $\mathfrak{D}$ is any small category, $\mathcal{C}$ is a category with a notion of weak equivalence. Then the functor category $\mathcal{C}^{\mathfrak{D}}$ has an obvious (objectwise) notion of weak equivalence. However, in general, it is not the case that the natural map $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}}) \to (\mathrm{Ho}(\mathcal{C}))^{\mathfrak{D}}$ is an equivalence of categories. This has some interest in the case where $\mathfrak{D}$ is a group (a category with one object and all morphisms invertible). In this case, $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}})$ is the equivariant homotopy theory of $\mathcal{C}$-objects with $\mathfrak{D}$ actions, while $\mathrm{Ho}(\mathcal{C})^{\mathfrak{D}}$ is the cateogry of $\mathcal{C}$-objects upon which $\mathfrak{D}$ "acts up to homotopy". Notice that the constant map $\mathfrak{D} \to \{e\}$ of $\mathfrak{D}$ to the trivial group yields in the obvious manner a functor k: $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^{\mathfrak{D}})$. In the case that $\mathcal{C}$ is either topological spaces or simplicial sets, k has a left adjoint B which takes a $\mathfrak{D}$-space $E\mathfrak{D} \times_{\mathfrak{D}} X$, where $E\mathfrak{D}$ is a contractible space upon which $\mathfrak{D}$ acts freely. In particular, B takes the terminal object of $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}})$ to the classifying space of $\mathfrak{D}$.

More generally, if $\mathfrak{D}$ is any category, a left adjoint to $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^{\mathfrak{D}})$ is called a homotopy colimit functor, and a right adjoint is called a homotopy limit functor. Notice that a left adjoint to $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})^{\mathfrak{D}}$ would be a colimit functor, and a right adjoint would be a limit functor. Thus, the difference between the homotopy colimit (resp. limit) and the ordinary colimit (resp. limit) is closely related to the extent to which $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}}) \to \mathrm{Ho}(\mathcal{C})^{\mathfrak{D}}$ fails to be an equivalence of categories.

More generally, if Φ: $\mathfrak{D} \to \mathcal{E}$ is any functor, there is an induced functor $\mathrm{Ho}(\Phi)^*$: $\mathrm{Ho}(\mathcal{C}^{\mathcal{E}}) \to \mathrm{Ho}(\mathcal{C}^{\mathfrak{D}})$ given by composition. A left adjoint to $\mathrm{Ho}(\Phi)^*$, if it exists, will be unique up to isomorphism, and will be called the "homotopy left Kan extension along Φ", and will be denoted by $\mathrm{Ho}\, L^{\Phi}$: $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}}) \to \mathrm{Ho}(\mathcal{C}^{\mathcal{E}})$. Similarly, a right adjoint $\mathrm{Ho}\, R_{\Phi}$ to $\mathrm{Ho}(\Phi)^*$ will be called the "homotopy right Kan extension along Φ."

At the present degree of generality, it is almost impossible to know whether or not a homotopy colimit will exist. However, as we shall see, with more structure, it is possible to show that homotopy colimits or limits exist, and even to show that they are induced by the colimit or limit functors defined on certain full subcategory.

A somewhat more general problem than the problem of whether or not homotopy colimits or homotopy limits exist is whether or not certain functors on the homotopy level have adjoints. The following result can be found in K. Brown's [AHT]:

ADJOINT FUNCTOR LEMMA. *Let* $S\colon \mathcal{C}_1 \to \mathcal{C}_2$ *be left adjoint to* $T\colon \mathcal{C}_2 \to \mathcal{C}_1$ *where* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are categories with a notion of weak equivalence. If S and T preserve weak equivalences, then* $\mathrm{Ho}(S)\colon \mathrm{Ho}(\mathcal{C}_1) \to \mathrm{Ho}(\mathcal{C}_2)$ *is left adjoint to* $\mathrm{Ho}(T)$, *where* $\mathrm{Ho}(S)$ *and* $\mathrm{Ho}(T)$ *are constructed from S and T by the universal property of localization.*

Unfortunately, in many examples, if S is left adjoint to T, while S may preserve weak equivalences, T may not. It is possible, however, that there is a full subcategory $\mathcal{C}_1'$ of $\mathcal{C}_1$ such that S takes its values in $\mathcal{C}_1'$, $\mathrm{Ho}(\mathcal{C}_1') \to \mathrm{Ho}(\mathcal{C}_1)$ is an equivalence of categories, and T preserves weak equivalences when restricted to $\mathcal{C}_1'$. Then T induces a functor from $\mathrm{Ho}(\mathcal{C}_1')$ to $\mathrm{Ho}(\mathcal{C}_2)$ which has an extension (which is unique up to isomorphism) to $\mathrm{Ho}(\mathcal{C}_1)$. We call this isomorphism $\mathrm{Ho}(T)$. Since it is clearly adjoint to $\mathrm{Ho}(S)$, it is unique up to isomorphism, and thus does not depend upon the choice of $\mathcal{C}_1'$. Thus we are led to search for such categories $\mathcal{C}_1'$ of "good" objects.

Ken Brown gives axioms for such "good" categories in [AHT]. We shall adopt most of his ideas while modifying the point of view to concentrate on finding "good" subcategories (for whatever problem is at hand) of a given category with a notion of weak equivalence. Also, he gives his axioms in terms of fibrations; our axioms are in terms of cofibrations and are roughly dual to his. Later, we shall consider Quillen's axioms for homotopy theory which include both cofibrations and fibrations in a closely related and dual manner.

By a left homotopy structure on a category $\mathcal{C}$ with a notion of weak equivalence, we mean a specification of certain morphisms as "cofibrations", so that the following axioms are valid:

(LH 0) $\mathcal{C}$ has finite colimits.

(LH 1) The composition of two cofibrations is a cofibration.

(LH 2) Every isomorphism is a cofibration.

(LH 3) (Mapping Cylinder Axiom) Every map f in $\mathcal{C}$ can be factored as $f = pi$, where i is a cofibration and p is a weak equivalence.

(LH 4) (Cobase Extension Axiom) If we have a cocartesian square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

if f is a cofibration, so is f'. If f is *also* a weak equivalence, so is f'.

There are two other axioms which are sometimes desirable, but which we will never assume without explicitly so stating.

(LH 5) (Homotopy Extension Axiom) Every cofibration which is also a weak equivalence has a left inverse.

Suppose that we have a well ordered system of objects and maps $\{A_n \to A_{n+1}\}$ in $\mathcal{C}$. Define A_n' to be the colimit over all ordinal $r < n$ of A_r. We call a map $\{A_n \to B_n\}$ of well ordered systems a weak equivalence if each $A_n \to B_n$ is a weak equivalence. We call it a cofibration if each $A_{n+1} \sqcup^{A_n} B_n' \to B_{n+1}$ is a cofibration. This defines a left homotopy structure on the category of direct systems of objects indexed by any given ordinal number.

> (LH 6) (Continuity Axiom) $\mathcal{C}$ has colimits. The colimit functor preserves cofibrations and it preserves weak equivalences between cofibrant objects for well ordered direct systems of objects.

We shall reserve the symbol ϕ for the initial object of $\mathcal{C}$, and the symbol $*$ for the terminal object if there is one. If A, B are two objects of $\mathcal{C}$, we write $A \sqcup B$ for their coproduct. If we have a diagram $B \leftarrow A \to C$, we write $B \sqcup^A C$ for the colimit of this diagram; this is called the coproduct of B with C under A.

By an interval on A we shall mean an object $I(A)$ together with a cofibration i: $A \sqcup A \to I(A)$ and a weak equivalence p: $I(A) \to A$ such that pi is the identity on each summand of $A \sqcup A$. If f, g: $A \to B$, by a homotopy from f to g we mean an interval $I(A)$ on A together with a map F: $I(A) \to B$ whose composition with i is f on the first summand, g on the second. Homotopic maps have the same image in $\mathrm{Ho}(\mathcal{C})$.

Many of the standard constructions of homotopy theory, such as composition of homotopies with maps of (path) composition of homotopies carry over certain well behaved objects, those objects A for which the initial map $\phi \to A$ is a cofibration. For $\mathcal{C}$ the category of simplicial sets or the category of topological spaces with weak equivalence equal to homotopy equivalence, the initial map $\phi \to A$ is always a cofibration. However, in the category of basepointed spaces, the initial map will be a cofibration only if the basepoint is nondegenerate. These and other examples will be discussed later.

We call an object A cofibrant if the initial map $\phi \to A$ is a cofibration. The following results follow from (LH 0)–(LH 4). Their proofs are left to the reader as enlightening exercises in abstract homotopy theory.

LEMMA 2.1. *If A, B are cofibrant, C, D are arbitrary, f: $A \to B$, g_0, g_1: $B \to C$, h: $C \to D$, then if g_0 is homotopic to g_1, $g_0 f$ is homotopic to $g_1 f$, and hg_0 is homotopic to hg_1. If g_2: $B \to C$ is homotopic to g_1, then g_0 is homotopic to g_2.*

LEMMA 2.2. *Let $\mathcal{C}_\phi$ be the full subcategory of $\mathcal{C}$ consisting of the cofibrant objects. Then* $\mathrm{Ho}(\mathcal{C}_\phi) \to \mathrm{Ho}(\mathcal{C})$ *is an equivalence of categories.*

LEMMA 2.3. *If A, B are cofibrant,* $\mathrm{Hom}(\gamma(A), \gamma(B))$ *is the direct limit of the set of homotopy classes of maps of A into objects B' which are equipped with maps $B \to B'$ which are cofibrations and weak equivalences. If the homotopy extension axiom is satisfied,* $\mathrm{Hom}(\gamma(A), \gamma(B))$ *is just the set of homotopy classes of maps from A to B.*

LEMMA 2.4 (EXCISION LEMMA). *If A, B, C are cofibrant, and if f: $A \to B$ is a cofibration, g: $A \to C$ is a weak equivalence, then the cobase extension g':*

$B \to B \sqcup^A C$ of g along f is a weak equivalence.

LEMMA 2.5 (PASTING LEMMA). *Given a commutative cube*

$$\begin{tikzcd}[row sep=small, column sep=small]
 & A \arrow[ld] \arrow[rr] \arrow[dd] & & B \arrow[ld] \arrow[dd] \\
A' \arrow[rr, crossing over] \arrow[dd] & & B' & \\
 & C \arrow[ld] \arrow[rr] & & D \arrow[ld] \\
C' \arrow[rr] & & D' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in which all objects are cofibrant and the front and back faces are cocartesian with all maps cofibrations, if $A \to A'$, $B \to B'$, $C \to C'$ are weak equivalences, so is $D \to D'$.

The Pasting lemma is useful for proving that the homotopy type of a suspension depends only on the homotopy type of the original space and for other theorems of this type.

The homotopy extension axiom holds for the category of topological spaces, where weak equivalences are homotopy equivalences, and cofibrations are closed embeddings with the homotopy extension property. This axiom fails for the category of simplicial sets. In general, if $\mathcal{C}$ is a category with a left homotopy structure, A, B are cofibrant objects, $\mathrm{Hom}(\gamma(A), \gamma(B))$ is the colimit of the set of homotopy classes of maps of A into objects C for which there is a specified map $B \to C$ which is both a cofibration and a weak equivalence (see Brown [AHT]). Thus, if the homotopy extension axiom holds, for all cofibrant A, B, $\mathrm{Hom}(\gamma(A), \gamma(B))$ is just the set of homotopy classes of maps from A to B. It is customary to write $[A, B]$ for $\mathrm{Hom}(\gamma(A), \gamma(B))$, and $\pi(A, B)$ for the set of homotopy classes of maps from A to B.

The axiom of continuity is useful for the construction of homotopy colimits and, more generally, the construction of homotopy direct images (or homotopy left Kan extensions).

Our axiomatization of left homotopy theory allows for a fairly economical axiomatization of homology and cohomology theories. By a cohomology functor on $\mathcal{C}$, we mean a set valued contravariant functor H which takes weak equivalences to isomorphisms, and which satisfies the following axioms:

(H 1) (Additivity) *If $\{A_\alpha\}$ is a family of cofibrant objects, $H(\sqcup\{A_\alpha\}) \to \sqcap\{H(A_\alpha)\}$ is an isomorphism.*

(H 2) (Excision) *If A is cofibrant, $A \to B$, $A \to C$ cofibrations, the map $H(B \sqcup^A C) \to H(A) \sqcap_{H(A)} H(C)$ is an epimorphism.*

If there is a cofibrant terminal object in $\mathcal{C}$, these axioms can be extended in the usual manner to cohomology functors H^{-n} for $n > 0$. This follows the lines of Dold [HEF]. Dual axioms characterize homology. For any object C in $\mathcal{C}$, $H(A) = [A, C]$ is a cohomology functor. If cohomology functors H^n exist for all n with $(H^n)^{-1} \cong H^{n-1}$ all n, we call the collection of cohomology functors $\{H^n\}$ and the given isomorphisms a cohomology theory.

There are various categories with left homotopy structures associated to a given category with a left homotopy structure. For example, if A is an object of $\mathcal{C}$, let $A/\mathcal{C}$ be the category of all diagrams $A \to B$ in $\mathcal{C}$. If $\mathcal{C}$ has a terminal object $*$, $*/\mathcal{C}$ is the category of basepointed objects over $\mathcal{C}$. If f: $A \to B$, g: $A \to C$, h: $B \to C$ with $g = hf$, we say that h is a cofibration of f to g if it is a cofibration of B to C in $\mathcal{C}$. Weak equivalences are defined similarly.

More interesting is the following. Suppose that $\mathcal{P}$ is a finite partially ordered set. If Φ, Ψ: $\mathcal{P} \to \mathcal{C}$, η: $\Phi \to \Psi$, call η a weak equivalence if for all $P \in \mathcal{P}$, $\eta(P)$: $\Phi(P) \to \Psi(P)$ is a weak equivalence. Call η a cofibration if for all P, the following map is a cofibration:

$$\operatorname{Colim}\{\Psi(P')|P' < P\} \sqcup_{\operatorname{Colim}\{\Phi(p')|P'<P\}} \Phi(P) \to \Psi(P). \tag{2.6}$$

Then the functor category $\mathcal{C}^{\mathcal{P}}$ can easily be seen to be a category with a left homotopy structure. The following follows from the pasting lemma and induction on the size of $\mathcal{P}$.

Lemma 2.7. *If Φ, Ψ: $\mathcal{P} \to \mathcal{C}$ are cofibrant, and η: $\Phi \to \Psi$ is a weak equivalence, then* $\operatorname{Colim}(\eta)$: $\operatorname{Colim}(\Phi) \to \operatorname{Colim}(\Psi)$ *is a weak equivalence.*

Corollary 2.8. *If $\mathcal{P}$ is a finite ordered set, the homotopy colimit functor exists on* $\operatorname{Ho}(\mathcal{C}^{\mathcal{P}})$.

If $\mathcal{C}$ satisfies the continuity axiom, the requirement that $\mathcal{P}$ be a finite ordered set in (2.8) is superfluous, and $\mathcal{P}$ can be replaced by any category. The proof, however, is somewhat indirect. We shall outline the proof, as it seems not to be obvious to us.

Theorem 2.9 (Homotopy Left Kan Extension Theorem). *If $\mathcal{C}$ is a category with a left homotopy structure which satisfies the continuity axiom, and if Φ: $\mathcal{A} \to \mathcal{B}$ is any map of categories, let* $\operatorname{Ho}(\Phi)^*$: $\operatorname{Ho}(\mathcal{C}^{\mathcal{B}}) \to \operatorname{Ho}(\mathcal{C}^{\mathcal{A}})$ *be the induced functor. Then* $\operatorname{Ho}(\Phi)^*$ *has a left adjoint* $\operatorname{Ho} L^{\Phi}$.

This theorem states that there is a homotopy theoretic left Kan extension along Φ given by $\operatorname{Ho} L^{\Phi}$. Since, if $\Psi\Phi$ is defined, $\operatorname{Ho}(\Psi\Phi)^* = \operatorname{Ho}(\Phi)^*\operatorname{Ho}(\Psi)^*$, we see that up to a natural isomorphism, $\operatorname{Ho} L^{\Psi\Phi} = \operatorname{Ho} L^{\Phi} H L^{\Psi}$. Notice that we do not claim that there is a left homotopy structure on the functor categories $\mathcal{C}^{\mathcal{B}}$ and $\mathcal{C}^{\mathcal{A}}$ in which $\operatorname{Ho} L^{\Phi}$ is just $\operatorname{Ho}(L^{\Phi})$, where L^{Φ} is left Kan extension along Φ. There is, however, a version of the Eilenberg-Moore spectral sequence for $\operatorname{Ho} L^{\Phi}$ despite the fact that this functor is not so explicitly defined. This spectral sequence has $E^2_{p,q} = L^{\Phi}_p(H_q(X))$ and converges to $H_{p+q}(\operatorname{Ho} L^{\Phi}(X))$, where H_* is an abelian group valued homotopy theory on $\mathcal{C}$, L^{Φ}_p is the pth left derived functor of left Kan extension along Φ, X is in $\operatorname{Ho}(\mathcal{C}^{\mathcal{A}})$, and H_* is extended to H_*: $\operatorname{Ho}(\mathcal{C}^{\mathcal{A}}) \to Ab_*^{\mathcal{A}}$ by objectionwise extension, where Ab_* is the category of graded abelian groups. In particular, if $\mathcal{A}$ is a group, $\mathcal{B}$ consists of a point, $\operatorname{Ho} L^{\Phi}$ is the homotopy theoretic quotient functor. In this case, we see that the Eilenberg-Moore spectral sequence has $E^2_{p,q} = H_p(\mathcal{A}, H_q(X))$ and converges to $H_{p+q}(X//\mathcal{A})$, where $-//\mathcal{A}$ is the homotopy theoretic quotient, X is an object of $\mathcal{C}$ with an $\mathcal{A}$-action, and $H_*(\mathcal{A}, -)$ denotes group homology. This is the form of the Eilenberg-Moore spectral sequence which is familiar to most people.

The proof of the homotopy left Kan extension theorem is based upon an analysis of a category in terms of its subdivisions. If $\mathcal{A}$ is a category, by $Sd(\mathcal{A})$ we mean the category whose objects consist of pairs (n, θ) where $n \geqslant 0$ is an integer, $\theta: \mathbf{n} \to \mathcal{A}$ is a functor, where $\mathbf{n}$ is the partially ordered set $\{0 \to 1 \to \cdots \to n\}$ regarded as a category. We do not allow those $(\theta, \mathbf{n})$ for which any $\theta(i) \to \theta(i+1)$ is an identity map. Morphisms are defined by letting $Sd(\mathcal{A})$ be the full subcategory with the objects described of the localization of the category of finite ordered sets over $\mathcal{A}$, where localization is taken with respect to epimorphisms of finite ordered sets.

(Notice that this is not the subdivision given by Mac Lane **[CWM]** which takes into account only $n = 0, 1$.) It is easy to see that $Sd^2(\mathcal{A})$ is always a partially ordered set. Further, it is a locally finite partially ordered set in the sense that every element has only a finite set of smaller elements. The definition of cofibrations in $\mathcal{C}^{\mathcal{P}}$ for a locally finite partially ordered set still makes sense and defines a left homotopy structure on $\mathcal{C}^{\mathcal{P}}$. Further, Lemmas (2.6) and (2.7) both hold if $\mathcal{C}$ satisfies the continuity axiom.

Let $\tau: Sd(\mathcal{A}) \to \mathcal{A}$ be defined by $\tau(n, \theta) = \theta(n)$. Then the category τ/A for A in $\mathcal{A}$ has as objects all diagrams $A_0 \to \cdots \to A_n \to A$. From this observation, the following lemma is not difficult to prove.

LEMMA 2.10. *If $\mathcal{K}$ is any category with colimits, the map $\mathcal{K}^\tau: \mathcal{K}^{\mathcal{A}} \to \mathcal{K}^{Sd(\mathcal{A})}$ has a left adjoint which is also a left inverse.*

The left adjoint to $\mathcal{K}^\tau$ is, of course, left Kan extension along τ. The previous lemma, together with the following lemma, will yield the subdivision corollary from which the homotopy direct image theorem follows easily.

LEMMA 2.11. *If $\mathcal{P}$ is a locally finite partially ordered set, $\mathfrak{D}$ any category, φ: $\mathcal{P} \to \mathfrak{D}$ a functor, then if $\mathcal{C}$ is a category with a left homotopy structure which satisfies the continuity axiom, the left Kan extension functor L^φ: $\mathcal{C}^{\mathcal{P}} \to \mathcal{C}^{\mathfrak{D}}$ takes weak equivalences between cofibrant objects to weak equivalences.*

COROLLARY 2.12 (SUBDIVISION COROLLARY). *If $\mathfrak{D}$ is any small category, $\mathcal{C}$ any category with a left homotopy structure satisfying the continuity axiom, then the functor $\mathrm{Ho}(\mathcal{C}^{\mathfrak{D}}) \to \mathrm{Ho}(\mathcal{C}^{Sd^2(\mathfrak{D})})$ has a left adjoint which is a left inverse. Thus it is a full and faithful functor. Thus the existence of homotopy colimits for $\mathcal{C}^{Sd^2(\mathfrak{D})}$ implies the existence of homotopy colimits for $\mathcal{C}^{\mathfrak{D}}$.*

If $\mathcal{O}$ is the category of finite ordered sets, let $s\mathcal{C}$ be the category of contravariant functors $\mathcal{O} \to \mathcal{C}$. We call $s\mathcal{C}$ the category of simplicial objects over $\mathcal{C}$. The homotopy colimit functor from $\mathrm{Ho}(s\mathcal{C})$ to $\mathrm{Ho}(\mathcal{C})$ is related to the geometric realization functor, when that functor exists. It is now a matter of calculation to show that there is, for any homology theory H_* on $\mathcal{C}$, a spectral sequence for $H_*(\mathrm{Ho\ colim}(X))$ which has $E^1_{p,q} = H_p(X(\mathbf{q}))$.

The opposite category to a category with a left homotopy structure is called a category with a right homotopy structure. The opposite to a weak equivalence is again called a weak equivalence, and the opposite to a cofibration is callled a fibration. The opposite to a cofibrant object is called a fibrant object. If $\mathcal{C}$ is a category with a left homotopy structure, the opposite category clearly has the property that $\mathrm{Ho}(\mathcal{C}^0) = \mathrm{Ho}(\mathcal{C})^0$.

If we have a commutative square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

in a category $\mathcal{C}$ with a right homotopy structure, W is called a homotopy theoretic fiber product of Y with X over Z if, in the category of squares over $\mathcal{C}$, this square is weakly equivalent to one which is homotopy cartesian. If W, X, Y, Z are all fibrant, the square will be weakly equivalent to a homotopy cartesian one if and only if for some (or equivalently, for all) factorizations $Y \to Y' \to Z$, where $Y \to Y'$ is a weak equivalence and $Y' \to Z$ is a fibration, $W \to Y' \sqcap_Z X$ is a weak equivalence.

If $\mathcal{C}$, $\mathfrak{D}$ are two categories with right homotopy structures satisfying the continuity axiom, we call a functor Φ: $\mathcal{C} \to \mathfrak{D}$ a homotopy continuous functor if it preserves fibrant objects, weak equivalences between fibrant objects, sequential limits of fibration up to weak equivalence, and homotopy fiber products. It is now possible to show that homotopy continuous functors preserve homotopy limits.

3. Further structures for homotopy theory. There are numerous extra structures which can be imposed upon a category with weak equivalences which will enrich its homotopy theory. The main ones which we shall discuss have to do with Quillen's axioms (as given in his **[RH]**) which deal with both left and right homotopy structures and the relationship between these. A subsidiary structure will have to do with "enriched" homotopy theory, where one has categories equipped with Hom-functors which take their values in some fixed category theory of such categories as the category of topological spaces upon which some *topological* group acts. Unfortunately, we know of no proofs that homotopy limits or homotopy colimits exist in general in such enriched settings, though in many individual cases (particularly when the "enriching" is in terms of simplicial sets), one can construct homotopy colimits and limits by special means. Indeed, I would expect that in the near future homotopy limits and colimits will be shown to exist whenever the appropriate enriched limits and colimits exist.

Quillen's axioms are self dual, in the sense that if they hold for a category, they hold for its opposite when the opposite of a cofibration is a fibration, etc. These axioms show that the weak equivalences and the fibrations (resp. the cofibrations), the cofibrations (resp. fibrations) are determined as those maps which admit solutions to certain "lifting problems." More specifically, if f: $A \to B$, g: $C \to D$ are two maps in $\mathcal{C}$, we say that f has the left lifting property (LLP) with respect to g if for all u: $A \to C$, v: $B \to D$ with $vf = gu$, there is w: $B \to C$ with $gw = v$, $wf = u$. If f has the LLP with respect to g, we also say that g has the right lifting property (RLP) with respect to f.

If f: $A \to B$, g: $C \to D$, we call g a retract of f if there are maps u: $A \to C$, v: $B \to D$, u': $C \to A$, v': $D \to B$ with $gu = vf$, $fu' = v'g$, and both uu' and vv' are the identity

The axioms given here are equivalent to those given by Quillen in **[HA]** for what is there called a "closed model category," while a "model category" satisfied slightly weaker axioms. Since there are no important examples (indeed, to my knowledge, there are no examples) of "model categories" which are not "closed model categories", I propose that the term "closed" be dropped, as the term "closed category" already has an accepted meaning. Later, I shall refer to "model closed categories" as model categories which are closed categories in the conventional sense and enjoy certain additional properties.

A model category is defined to be a category $\mathcal{C}$ which has a notion of weak equivalence, and in which two classes of morphisms have been designated as cofibrations and as fibrations, respectively, such that the following axioms hold:

M0 $\mathcal{C}$ has finite limits and finite colimits.

M1 (Retraction axiom) Every retract of a cofibration, weak equivalence, or fibration, respectively, is again a cofibration, weak equivalence, or fibration, respectively.

M2 (Lifting axiom) Every cofibration has the LLP with respect to every fibration which is also a weak equivalence. Every fibration has the RLP with respect to every cofibration which is also a weak equivalence.

M3 (Factorization) Every map f can be factored as $f = pi = qj$, where i, j are cofibrations, p, q are fibrations, and i and q are weak equivalences.

Notice that the factorization axiom for a model category is much stronger than the combination of the factorization axioms for left and right homotopy structures. The usual projection of a mapping cylinder (for a map of topological spaces) onto its base is a quasifibration, but not a fibration. However, Strom shows that with the usual cofibrations and with weak equivalences taken to be homotopy equivalences, the axioms above are all satisfied by the category of topological spaces when the fibrations are taken to be the Hurewicz fibrations (see Strom **[HCHC]**).

Notice the complete duality between cofibrations and fibrations in these axioms. Quillen shows in **[RH]** that the fibrations and cofibrations determine one another. If $\mathfrak{M}$ is a class of morphisms in a category, let LLP($\mathfrak{M}$) be the class of morphisms which have LLP with respect to all elements of $\mathfrak{M}$, and let RLP($\mathfrak{M}$) be analogous. A fibration or cofibration is called trivial if it is also a weak equivalence. What Quillen proved can be summarized as follows:

(a) LLP (fibrations) = trivial cofibrations,
(b) LLP (trivial fibrations) = cofibrations,
(c) RLP (cofibrations) = trivial fibrations,
(d) RLP (trivial cofibrations) = fibrations.

Quillen also showed that a morphism f in $\mathcal{C}$ is a weak equivalence if and only if its image $\gamma(f)$ in Ho($\mathcal{C}$) is an isomorphism. Finally, he shows that our axioms for both a left and right homotopy structure are satisfied.

There are several technical points which are worth noting about model categories.

LEMMA 3.1. *If A is cofibrant, X fibrant, then the left homotopy classes of maps* *from A to X agrees with the right homotopy classes, and both sets equal* $[A, X]$.

LEMMA 3.2. *If A is cofibrant, X fibrant, $A \to B$ a trivial cofibration, every morphism $A \to X$ extends to B.*

LEMMA 3.3. *If A is cofibrant, X fibrant, f, g: $A \to X$, F: $I(A) \to X$ a left homotopy from f to g, then for any other interval $J(A)$ on A, there is a left homotopy G; $J(A) \to X$ from f to g.*

If π: $X \to Y$, and F: $I(A) \to X$ is a left homotopy, we say that F is a vertical homotopy if $\pi F = fp$, where p: $I(A) \to A$ is the projection, and f: $A \to Y$ is some map.

LEMMA 3.4. *If f: $A \to B$ is a trivial fibration between cofibrant objects, there is g: $B \to A$ with $fg = 1_B$, and such that gf is left homotopic to 1_A by a vertical homotopy.*

Notice that (3.4) states that any map between cofibrant objects factors into a cofibration followed by a fibration with a section which is a left vertical homotopy equivalence. If the map is a weak equivalence, the cofibration in the factorization is, of course, trivial. It was pointed out to me by my student, Chris Reedy, that (3.4) can be used to prove the only difficult part of the Continuity Axiom for model categories which have colimits. Thus the Continuity Axiom is, for model categories, just a matter of the existence of colimits.

The categories of simplicial sets, pointed simplicial sets, compactly generated topological spaces, and pointed compactly generated topological spaces are all closed categories in the sense that they possess "internal Hom-functors" and a related symmetric monoidal operation (Cartesian product or smash product). It is possible to show that the homotopy categories are also closed in this sense. To do this requires some further work.

Recall from Mac Lane [CWM] that a closed category $\mathcal{C}$ is one which possesses two functors $-\otimes-$: $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, HOM: $\mathcal{C}^0 \times \mathcal{C} \to \mathcal{C}$, such that $\otimes$ defines symmetric monoidal structure with a unit (which we write E) on $\mathcal{C}$, and such that there is a natural equivalence for all A, B, C: $\mathrm{Hom}(A, \mathrm{HOM}(B, C)) = \mathrm{HOM}(A \otimes B, C)$. Notice that $\mathrm{Hom}(A, B) = \mathrm{Hom}(E, \mathrm{HOM}(A, B))$ for all A, B.

A model category which is a closed category will be called a model closed category if either of the following (equivalent) axioms holds:

> (MC1) Given a cofibration i: $A \to B$ and a fibration p: $X \to Y$, the following map is a fibration. It is also a weak equivalence if either i or p is:
>
> $$\mathrm{HOM}(B, X) \to \mathrm{HOM}(A, X) \sqcap_{\mathrm{HOM}(A, Y)} \mathrm{HOM}(B, Y).$$
>
> (MC1*) Given cofibrations i: $A \to B$, j: $C \to D$, the following map is a cofibration. It is also a weak equivalence if either i or j is:
>
> $$(A \otimes D) \sqcup^{(A \otimes C)} (B \otimes C) \to B \otimes D.$$

Choosing cofibrant representatives for each isomorphism class in $\mathrm{Ho}(\mathcal{C})$, it is easy to see that $\otimes$ can be defined on $\mathrm{Ho}(\mathcal{C})$, and up to a unique isomorphism, does not depend upon the choices of representatives. Further, choosing fibrant representatives, we can define HOM on $\mathrm{Ho}(\mathcal{C})$ so that $\mathrm{Ho}(\mathcal{C})$ is a closed category. For example, this means that in the homotopy category of pointed simplicial sets or in the homotopy theory of pointed compactly generated spaces, the smash product of homotopy types and the pointed function complex (resp. function space) between homotopy types is well defined and enjoy the usual adjointness properties: $[A \wedge B, C] = [A, C^B]$ (here by homotopy type, I mean isomorphism classes in the homotopy category; I assume that representatives are chosen in each class).

Finally, if $\mathcal{C}$ is a model closed category, it can provide an "enrichment" for other model categories. For example, if G is a topological group (all spaces are assumed compactly generated), in the category of spaces with G actions, the Hom-sets naturally have topologies, and so are themselves topological spaces. We wish for this "enrichment" to be passed along to the homotopy categories. The following discussion will show how to do this.

If $\mathcal{C}$ is just a closed category, by a $\mathcal{C}$-enrichment of a category $\mathfrak{M}$ we mean a functor (which, by abuse of notation, we write as HOM) HOM: $\mathfrak{M}^0 \times \mathfrak{M} \to \mathcal{C}$, together with a naturally associative pairing $\mathrm{HOM}(K, L) \otimes \mathrm{HOM}(L, M) \to \mathrm{HOM}(K, M)$ and a unit $E \to \mathrm{HOM}(K, K)$ for all K, L, M in $\mathfrak{M}$. The enriched category $\mathfrak{M}$ is said to be tensored if there is a naturally associative functor $-\otimes-: \mathcal{C} \times \mathfrak{M} \to \mathfrak{M}$ and a natural isomorphism $\mathrm{Hom}(C, \mathrm{HOM}(K, L)) = \mathrm{Hom}(C \otimes K, L)$. The enriched category $\mathfrak{M}$ is said to be cotensored if there is a functor HOM: $\mathcal{C}^0 \times \mathfrak{M} \to \mathfrak{M}$ such that there is a naturally associative isomorphism $\mathrm{HOM}(K, \mathrm{HOM}(A, L)) \cong \mathrm{HOM}(A, \mathrm{HOM}(K, L))$.

We call a model category $\mathfrak{M}$ a $\mathcal{C}$-enriched model category (or simply a $\mathcal{C}$-model category) for a model closed category $\mathcal{C}$ if it is $\mathcal{C}$-enriched, tensored, and cotensored, and such that either of the following axioms holds.

(EM1) If $i: A \to B$ is a cofibration in $\mathcal{C}$, $p: M \to N$ is a fibration in $\mathfrak{M}$, the following is a fibration which is a weak equivalence if p is:

$$\mathrm{HOM}(B, M) \to \mathrm{HOM}(B, N) \times_{\mathrm{HOM}(A,N)} \mathrm{HOM}(A, M).$$

(EM1*) If $i: A \to B$ is a cofibration in $\mathcal{C}$, $j: K \to L$ is a cofibration in $\mathfrak{M}$, then the following is a cofibration which is a weak equivalence if j is:

$$(A \otimes L) \cup^{(A \otimes K)} (B \otimes K) \to B \otimes L.$$

In this case, it is not difficult to show that if $j: K \to L$ is a cofibration in $\mathfrak{M}$, $p: M \to N$ is a fibration in $\mathfrak{M}$, then the following is a fibration in $\mathcal{C}$ and a weak equivalence, whenever j or p is a weak equivalence:

$$\mathrm{HOM}(L, M) \to \mathrm{HOM}(L, N) \times_{\mathrm{HOM}(K,N)} \mathrm{HOM}(K, M).$$

It is possible to have a stronger enrichment structure. We say that a $\mathcal{C}$-enriched model category $\mathfrak{M}$ is strongly $\mathcal{C}$-enriched if in addition the map in (EM1) (or equivalently (EM1*)) is a weak equivalence whenever i is also a weak equivalence. In this case, $\mathrm{Ho}(\mathfrak{M})$ is a tensored and cotensored $\mathrm{Hom}(\mathcal{C})$-enriched category.

4. Simplicial objects and their homotopy theory. Let $\mathcal{O}$ be the category whose objects are the finite ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ for $n \geqslant 0$. Let $\mathcal{O}_n$ be the full subcategory of those $\mathbf{m}$ for $m \leqslant n$.

If $\mathcal{C}$ is any category, a simplicial object over $\mathcal{C}$ is just a contravariant functor $X\colon \mathcal{O} \to \mathcal{C}$. Notice that in this terminology, a simplicial set is a "simplicial object over the category of sets." A "simplicial object up to a dimension n over $\mathcal{C}$" is defined to be a contravariant functor $X\colon \mathcal{O}_n \to \mathcal{C}$. We write $s\mathcal{C}$ for the functor category of simplicial objects over $\mathcal{C}$, and $s_n\mathcal{C}$ for the functor category of simplicial objects up to degree n over $\mathcal{C}$. The restriction of functors gives us a functor $\rho_n\colon s\mathcal{C} \to s_n\mathcal{C}$. From the general abstract nonsense of Kan extensions, if $\mathcal{C}$ has finite colimits, ρ_n has a left adjoint σ_n, and if $\mathcal{C}$ has finite limits, ρ_n has a right adjoint σ^n. The composition $\sigma_n\rho_n$: $s\mathcal{C} \to s\mathcal{C}$ is called the n-skeleton functor, and is denoted by sk_n. The composition $\sigma^n\rho_n$ is called the n-coskeleton functor and is denoted by $c\ \mathrm{sk}^n$. These are determined by the property that for all simplicial objects X, Y, maps $X|\mathcal{O}_n \to Y|\mathcal{O}_n$ are in a 1-1 correspondence with maps $\mathrm{sk}_n X \to Y$ and with maps $X \to c\ \mathrm{sk}^n Y$. For simplicial sets X, $(\mathrm{sk}_n X)_m$ consists of those m-simplices which are the image under a degeneracy of a simplex of dimension $\leqslant n$. It is more difficult to describe $(c\ \mathrm{sk}^n X)_m$, though when $n = 0$, it is just the mth power of X_0.

If $\mathcal{C}$ has finite colimits, $s\mathcal{C}$ is tensored over the category of finite simplicial sets. The tensor construction is given as follows. If X is a simplicial object over $\mathcal{C}$, and if A is a finite simplicial set, let $(A \otimes X)_n$ be the coproduct of one copy X_n^α of X_n for each $\alpha \in A_n$. Face and degeneracy maps act by their separate action on both X and A. If $\mathcal{C}$ has finite limits, $\mathcal{C}$ is cotensored over the category of finite simplicial sets. This is less easy to show than the previous case, but it is not deep. The simplest way to construct $\mathrm{HOM}(A, X)$ is as the right Kan extension along the Yoneda functor of X, evaluated at A. In either case, $s\mathcal{C}$ is enriched by simplicial sets if we let $\mathrm{HOM}(X, Y)_n$ be $\mathrm{Hom}(\Delta^n \otimes X, Y)$ or $\mathrm{Hom}(X, \mathrm{HOM}(\Delta^n, Y))$ respectively, and if both finite limits and colimits exist, these two definitions of $\mathrm{HOM}(X, Y)$ will agree. This follows from general nonsense involving the Yoneda lemma, and is left to the reader.

The following result, so far as I know, is due to Reedy. His proof is quite ingenious, but it is too long to give here. It is curious that his proofs use both the cofibrations and the fibrations in essential ways, which has kept me from being able to extend them to prove similar results about categories with just a left homotopy structure. Indeed, I do not know whether $s\mathcal{C}$ will have a left homotopy structure if $\mathcal{C}$ does.

THEOREM 4.1. *If $\mathcal{C}$ is a model category, $s\mathcal{C}$ and each $s_n\mathcal{C}$ will have the structure of a model category which is tensored and cotensored over finite simplicial sets so that* EM1* *and* EM1 *are satisfied. If $\mathcal{C}$ has colimits and limits, $s\mathcal{C}$ and each $s\mathcal{C}$ are simplicial set model categories.*

The choice of the cofibrations which Reedy makes is as follows: $f\colon X \to Y$

is a cobifration if for all m, $\mathrm{sk}_m(Y)_{m+1} \sqcup^{\mathrm{sk}_m(X)_{m+1}} X_{m+1} \to Y_{m+1}$ is a cofibration. Fibrations are defined in a dual manner using coskeletons.

If $\mathcal{C}$ is a simplicial set model category with colimits, one can define a geometric realization functor R: $s\mathcal{C} \to \mathcal{C}$ as follows. Let $R(X)$ be the quotient of the disjoint union of all $\Delta^n \otimes X_m$ by the relation which, for all θ: $\Delta^n \to \Delta^m$, coequalizes the pair of maps $\Delta^m \otimes X_m \leftarrow \Delta^n \otimes X_m \to \Delta^n \otimes X_n$ which are $\theta \otimes X_m$ in the first case and $\Delta^n \otimes X(\theta)$ in the second. The functor R has a right adjoint Sing: $s\mathcal{C} \to \mathcal{C}$ given by $\mathrm{Sing}(Y)_n = \mathrm{HOM}(\Delta^n, Y)$. The following is due to Reedy, and is proved by careful application of the Pasting lemma.

THEOREM 4.2. *R: $s\mathcal{C} \to \mathcal{C}$ preserves cofibrations and weak equivalences between cofibrant objects.*

Notice that since Sing preserves fibration and weak equivalences between fibrant objects, R and Sing determine an adjoint pair of functors on the homotopy categories. If $\mathcal{C}$ is a strongly simplicial set enriched model category, for all fibrant Y we have $\mathrm{HOM}(\Delta^0, Y) = Y$, and $\mathrm{HOM}(\Delta^0, Y) \to \mathrm{HOM}(\Delta^n, Y)$ is always a weak equivalence. Thus Ho(Sing) is naturally equivalent to the constant assignment functor k, where $k(Y)_n = Y$. Thus $\mathrm{Ho}(R)$ is a homotopy colimit functor.

There is an instance where the geometric realization functor has a particularly simple form. Since $s\mathfrak{S}$, the category of simplicial sets, is a model closed category, we have the realization functor R: $ss\mathfrak{S} \to s\mathfrak{S}$. We have a second functor, D: $ss\mathfrak{S} \to s\mathfrak{S}$ given by the diagonal: $D(X)_n = (X_n)_n$. I claim that these are the same functor. To do this requires showing that D is left adjoint to Sing.

First, we observe that the Yoneda elements in $ss\mathfrak{S}$ are the $\Delta^{m,n}$ given by $(\Delta^{m,n})_{i,j} = (\Delta^m_i) \times (\Delta^n_j)$. Notice that for all X, $\mathrm{Hom}(\Delta^{m,n}, X) = X_{m,n}$. Thus, $\mathrm{Hom}(\Delta^{m,n}, \mathrm{Sing}(Y)) = \mathrm{HOM}(\Delta^n, Y)_m = \mathrm{Hom}(\Delta^m \times \Delta^n, Y)$. However, $D(\Delta^{m,n}) = \Delta^m \times \Delta^n$, so $R(\Delta^{m,n}) = D(\Delta^{m,n})$ for all m, n. However, every object of $ss\mathfrak{S}$ is the colimit of a diagram of the $\Delta^{m,n}$'s, so $D = R$.

5. Groupoids. A groupoid is nothing more than a category in which every map is an isomorphism. These arise in topology from the fundamental groupoid construction which assigns to a space X (or a simplicial set X) the groupoid $\Pi_1(X)$ whose objects are the points of X (resp. the 0-simplices of X) and whose morphisms between any two points (resp. 0-simplices) is the set of homotopy classes of paths between them. Associated to the groupoid $\Pi_1(X)$ is the collection $\pi_n(X)$ of group valued functors on $\Pi_1(X)$, where for a given object x of $\Pi_1(X)$, $\pi_n(X)(x)$ is the nth homotopy group $\pi_n(X, x)$ based at x. It is well known that every path from x_0 to x_1 gives a homomorphism $\pi_n(X, x_0) \to \pi_n(X, x_1)$ for all $n \geqslant 1$ (if $n = 0$, this is also defined, though we now always have the identity map, and $\pi_0(X)$ takes its values in the category of pointed sets). This gives us a group valued functor $\pi_n(X)$ on $\Pi_1(X)$. If f: $X \to Y$ is a weak equivalence of either spaces or simplicial sets, f_*: $\Pi_1(X) \to \Pi_1(Y)$ is an equivalence of categories, and this equivalence extends to the functors π_n.

If X is a simplicial set, there is a well-known theorem which states that

$\Pi_1(X)$ is the free groupoid generated by the 0-and 1-simplices modulo the relations $(d_0\sigma)(d_2\sigma) = d_1\sigma$ for each 2-simplex σ of X. In particular, if we look at $\Pi_1(\Delta^n)$, we see that it is just the complete localization of the partially ordered set $\mathbf{n} = \{0, \ldots, n\}$. Thus, since the classifying space $B\mathcal{G}$ for a category $\mathcal{G}$ is defined to be the simplicial set with $(B\mathcal{G})_n = \mathrm{Hom}(\mathbf{n}, \mathcal{G})$, we see that for all n, $\mathrm{Hom}(\Delta^n, B\mathcal{G}) = \mathrm{Hom}(\Pi_1(\Delta^n), \mathcal{G})$ if $\mathcal{G}$ is a groupoid. More generally, the result mentioned above implies that for all X, $\mathrm{Hom}(X, B\mathcal{G}) = \mathrm{Hom}(\Pi_1(X), \mathcal{G})$, and that thus Π_1 is left adjoint to B (*not* in the category of all small categories, but just the category of small groupoids). Indeed, the adjointness of Π_1 and B is equivalent to the description of $\Pi_1(X)$ given in terms of low dimensional simplices. Notice that the 0-simplices of $B\mathcal{G}$ are just the objects of $\mathcal{G}$. Thus, the map $\mathcal{G} \to \Pi_1 B\mathcal{G}$ is an isomorphism, so Π_1 is both left adjoint and left inverse to B.

One of the remarkable features of the category of groupoids is that it admits a model structure. The weak equivalences are the natural equivalences of categories. The cofibrations are the functors which are monomorphisms on objects. The fibrations are the functors which are carried to fibrations of simplicial sets by the classifying space construction. Every groupoid is both cofibrant and fibrant, so the homotopy theory is particularly simple. If G, H are two groups, $[G, H]$ is just the quotient of the set of group maps from G to H modulo the relation of being conjugate under the action of some element of H. Since every groupoid is naturally isomorphic to a disjoint union of groups, it is very easy to calculate the set of homotopy classes of maps between two groupoids.

Notice that both Π_1 and B preserve weak equivalences, and so directly determine a pair of adjoint functors on the homotopy theory. Further, Π_1 clearly preserves cofibrations and colimits, and B preserves fibrations and limits.

A slightly deeper fact is that Π_1 preserves fibrations. Indeed, notice that if $\varphi\colon \mathcal{G} \to \mathcal{H}$ is a map of groupoids, φ is a fibration if and only if for each g in $\mathcal{G}$, h in $\mathcal{H}$, and $\alpha\colon \varphi(g) \to h$, there is a g' in $\mathcal{G}$ and an $\alpha'\colon g \to g'$ with $\varphi(\alpha') = \alpha$ and thus $\varphi(g') = g$. Fibrations between groupoids needn't be epimorphic on objects, but if an element of $\mathcal{H}$ is in the image of a fibration φ, so are all isomorphic elements. Also, except when the target is a disjoint union of groups, a fibration needn't be full. However, every full functor is clearly a fibration if it is epimorphic on objects.

Let $\Pi_\infty(X)$ be the "wreath product" of $\Pi_1(X)$ with the direct product of all $\pi_n(x)$ for $n \geqslant 2$. That is, $\Pi_\infty(X)$ has the same objects as X, but $\mathrm{Hom}_\infty(x, y)$ consists of pairs (α, λ) where $\alpha \in \mathrm{Hom}(x, y)$ and $\lambda \in \Pi\{\pi_n(X, y) | n \geqslant 2\}$. Composition is defined by $(\beta, \mu)(\alpha, \lambda) = (\beta\alpha, \mu + \beta_*(\lambda))$, where β_* is the action of the path β on higher homotopy. If X has a single vertex x, then $\Pi_\infty(X)$ is just the wreath product of $\Pi_1(X, x)$ with the product of the higher homotopy groups. If X is simple, it is just the product of all the homotopy groups of X. Notice that if $p\colon X \to Y$ is a map of spaces, $\Pi_\infty(p)\colon \Pi_\infty(X) \to \Pi_\infty(Y)$ will be a fibration of groupoids if $\Pi_1(p)$ is a fibration and if for every vertex x of X, the maps $\pi_n(X, x) \to \pi_n(Y, p(x))$ are epimorphic for all $n \geqslant 2$. The following is a standard result in homotopy theory, rephrased in terms of groupoids. It holds for both spaces and simplicial sets.

LEMMA 5.1. *If p: $X \to Y$ is a fibration, q: $Z \to Y$ is any map, the map*

$$\Pi_\infty(X \sqcap_Y Z) \to \Pi_\infty(X) \sqcap_{\Pi_\infty(Y)} \Pi_\infty(Z)$$

is always an epimorphism. It is an isomorphism if $\Pi_\infty(p)$ is full (that is, p_*: $\pi_n(X, x) \to \pi_n(Y, p(x))$ is epimorphic for all x and all $n \geqslant 1$).

We will refer to a fibration p: $X \to Y$ of spaces or of simplicial sets as a full fibration if $\Pi_\infty(p)$ is full. Notice that the class of full fibrations is closed under composition, base extension (pull back), and retraction. Furthermore, any two fibrations which are weakly equivalent have the property that they are either both full or neither is full. Thus one may, without ambiguity, call a map f: $X \to Y$ a full map if, whenever it is factored as $f = pi$, p a fibration and i a weak equivalence, p is full, and the property of being a full map depends only on the isomorphism class of f in the homotopy category. Finally, if $f = gh$ and f is full, so is g.

If p: $\mathcal{G} \to \mathcal{H}$ is a map of simplicial groupoids, p will be a fibration (of simplicial groupoids) if and only if for all trivial cofibrations $K \to L$ of finite simplicial sets, the following map is a fibration of groupoids:

$$\mathrm{HOM}(L, \mathcal{G})_0 \to \mathrm{HOM}(K, \mathcal{G})_0 \sqcap_{\mathrm{HOM}(K,\mathcal{H})_0} \mathrm{HOM}(L, \mathcal{H})_0.$$

If this map is also full and $\mathcal{H}$ is fibrant, we shall refer to p as a full fibration of simplicial groupoids. If $\mathcal{G}$ and $\mathcal{H}$ are simplicial groups, the condition is equivalent to the condition that on the underlying simplicial sets, p is a Kan fibration. It is a result of John Moore (see May [**SOAT**]) that a map of simplicial groups will be a Kan fibration if it is an epimorphism.

More generally, we refer to a map p: $\mathcal{G} \to \mathcal{H}$ of simplicial groupoids as being full if for some weak equivalence $\mathcal{H} \to \mathcal{H}'$ with $\mathcal{H}'$ fibrant and some factorization of $\mathcal{G} \to \mathcal{H}'$ into a weak equivalence $\mathcal{G} \to \mathcal{G}'$ and a fibration p': $\mathcal{G}' \to \mathcal{H}'$, p' is full. As usual, if this is true for some choice of $\mathcal{G}'$, $\mathcal{H}'$, p', it is true for all choices.

Finally, we refer to a map f: $X \to Y$ of simplicial spaces or of bisimplicial sets as a full fibration if $\Pi_\infty(f)$ is full. Notice that this is a property which depends only on the weak equivalence class of f. We call a simplicial space or simplicial set fully fibrant if the map to the terminal object is a full fibration.

THEOREM 5.2. *If Y is fully fibrant, and if K is any weakly contractible finite simplicial set, the map $\Pi_\infty(\mathrm{HOM}(K, Y))_0 \to \mathrm{HOM}(K, \Pi_\infty(Y))_0$ is an isomorphism.*

PROOF. This is always true for K a simplex. We proceed by double induction on the dimension of K and on the number of nondegenerate simplices in K. Since any retract of an isomorphism is an isomorphism, if the theorem holds for K, it holds for any retract of K.

Next, suppose that the theorem holds for weakly contractible simplicial sets A, B, C, and $K = B \sqcup^A C$, where $A \to B$ is a cofibration, $A \to C$ is arbitrary. Since Y is fully fibrant, the map $\mathrm{HOM}(B, \Pi_\infty(Y))_0 \to \mathrm{HOM}(A, \Pi_\infty(Y))_0$ is a full fibration, so inductively, so is $\mathrm{HOM}(B, Y)_0 \to \mathrm{HOM}(A, Y)_0$. Thus by the Pasting lemma, the theorem is true for K.

We have proved that the class of those K for which the theorem holds includes the simplices and is closed under retraction and under the attachment of simplices along weakly contractible subcomplexes. Thus the class includes all weakly contractible finite simplicial sets.

COROLLARY 5.3. *If Y is a fully fibrant simplicial space or bisimplicial set, and* Ex *is Kan's extension functor,* $\Pi_\infty(\mathrm{Ex}(Y)) = \mathrm{Ex}(\Pi_\infty(Y))$.

PROOF. Recall that $\mathrm{Ex}(Y)_n = \mathrm{HOM}(Sd(\Delta^n), Y)_0$. However, the subdivision of Δ^n is again a finite simplicial set which is weakly contractible.

6. Realization and fibrations. The category $ss\mathfrak{S}$ of bisimplicial sets has a curious technical advantage for us over the category of simplicial spaces. The realization functor R: $ss\mathfrak{S} \to s\mathfrak{S}$ has a left adjoint Q: $s\mathfrak{S} \to ss\mathfrak{S}$. Thus, if we have a map f: $X \to Y$ of bisimplicial sets, and we wish to know whether or not $R(f)$ is a fibration, we merely have to verify that f has the RLP with respect to the image under Q of every trivial cofibration. This will be our approach.

If X is a bisimplicial set, we write X_n for the simplicial set $(X_n)_m = X_{nm}$. Then $X_n = \mathrm{HOM}(\Delta^n, X)_0$. We will write, for a simplicial set K, $\mathrm{Hom}(K, X)$ for $\mathrm{HOM}(K, X)_0$. Recall that f: $X \to Y$ is a fibration in $ss\mathfrak{S}$ if and only if for all cofibrations i: $A \to B$ of finite simplicial sets, the map

$$\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y) \tag{6.1}$$

is a fibration of simplicial sets. We will call f an epifibration if for all trivial cofibrations i: $A \to B$ of finite simplicial sets, the map (6.1) is an epimorphism. Since the map is a fibration, it suffices that the map be an epimorphism on components. Our main technical result (which requires the promised ad hoc argument) is the following:

THEOREM 6.2. *If f: $X \to Y$ is an epifibration of bisimplicial sets, $R(f)$: $R(X) \to R(Y)$ is a fibration.*

We call a map with the LLP with respect to all epifibrations an epicofibration. Since an epifibration is a fibration, every trivial cofibration is an epicofibration. Thus if i: $A \to B$ is a cofibration of simplicial sets, j: $K \to L$ is a trivial cofibration, the following map is an epicofibration: $(A \times L) \cup^{(A \times K)} (B \times K) \to B \times L$.

A fibration is an epimorphism if and only if it has the RLP with respect to all $\phi \to \Delta^n$. Thus, since the inclusion of a vertex in a weakly contractible ("trivial") simplicial set is a trivial cofibration, a fibration of simplicial sets is an epimorphism if and only if it has the RLP with respect to all $\phi \to T$ for T trivial. Thus for all cofibrations i: $A \to B$ and all trivial T, $A \times T \to B \times T$ is an epicofibration.

To begin the proof of (6.2), we must investigate the functor Q. The existence of Q is automatic, as it is simply the left Kan extension of functors along the diagonal $\mathcal{O} \to \mathcal{O} \times \mathcal{O}$. On a simplex Δ^n, we have automatically, from the Yoneda lemma, that $Q(\Delta^n) = \Delta^{n,n}$. Since Q preserves colimits, and since every simplicial set is the colimit of a diagram of simplices, this describes Q completely. If A, B are simplicial sets, let $A \times B$ be the bisimplicial set given by $(A \times B)_{m,n} = A_m \times B_n$. Then $R(A \times B) = A \mathbin{\bar{\times}} B$, so the diagonal map $A \to A \mathbin{\bar{\times}} A$ defines for us, by adjointness, a map $Q(A) \to A \times A$.

In order to prove (6.2), it suffices to show that Kan's lifting criterion is met by $R(f)$. This states that when $\Lambda \subset \Delta^n$ is the inclusion of all but one face of dimension $n-1$, the map $\Lambda \to \Delta^n$ has LLP with respect to $R(f)$, or, equivalently, that $Q(\Lambda) \to Q(\Delta^n)$ is an epicofibration. Now the map $Q(\Lambda) \to Q(\Delta^n) = \Delta^n \times \Delta^n$ factors through the epicofibration $\Lambda \times \Lambda \to \Delta^n \times \Delta^n$, so it suffices to show that $Q(\Lambda) \to \Lambda \times \Lambda$ is an epicofibration. We first show that it is a monomorphism, and then show that $\Lambda \times \Lambda$ is obtained from $Q(\Lambda)$ by repeated cobase extensions of epicofibrations.

It is not always the case that the map $Q(A) \to A \times A$ is a monomorphism. A counterexample is given by letting A be the quotient of Δ^2 obtained by collapsing a face.

LEMMA 6.3. *If A is a subcomplex of Δ^n for some n, $Q(A) \to A \times A$ is a monomorphism.*

To prove this lemma, we need some way to show that a map of bisimplicial sets is a monomorphism. The next lemma gives us such a criterion.

LEMMA 6.4. *If $f\colon X \to Y$ is a map of bisimplicial sets, f is a monomorphism if and only if $R(f)$ is.*

PROOF. If f is a monomorphism, clearly so is its restriction to the diagonal. If m, n are arbitrary, there exists a k, together with monomorphisms $\delta'\colon \mathbf{m} \to \mathbf{k}$, $\delta''\colon \mathbf{n} \to \mathbf{k}$ and epimorphisms $\sigma'\colon \mathbf{k} \to \mathbf{m}$, $\sigma''\colon \mathbf{k} \to \mathbf{n}$ with $\sigma'\delta'$ and $\sigma''\delta''$ the identity. If $R(f)$ is a monomorphism, $X(\sigma', \sigma'')\colon X_{n,n} \to X_{k,k}$ has a left inverse, and thus is a monomorphism. Since $f_{k,k}\colon X_{k,k} \to Y_{k,k}$ is a monomorphism, so is $f_{m,n}\colon X_{m,n} \to Y_{m,n}$.

Suppose that A is a subcomplex of a simplex Δ^n. Then every nondegenerate simplex $\Delta^m \to A$ of A is a face of Δ^n, and thus a retract of A. Thus $Q(\Delta^m) \to Q(A)$ is a monomorphism as is $RQ(\Delta^m) \to RQ(A)$. Suppose $T_1, T_2 \to A$ are two nondegenerate simplices of A, $T = T_1 \sqcap_A T_2$. Then T is either empty or it is another nondegenerate simplex of A. Notice that also we have $T = T_1 \sqcap_{\Delta^n} T_2$, since the fiber product of two subcomplexes is their intersection. However, $RQ(T_1) \sqcap_{RQ(\Delta^n)} RQ(T_2) = (T_1 \sqcap T_1) \sqcap_{(\Delta^n) \sqcap (\Delta^n)} (T_2 \sqcap T_2) = T \sqcap T = RQ(T)$. Thus the intersection in $RQ(\Delta^n)$ of $RQ(T_1)$ and $RQ(T_2)$ is $RQ(T)$. Thus the intersection of $RQ(T_1)$ with $RQ(T_2)$ in A, which includes $RQ(T)$, cannot be any larger than the intersection in $RQ(\Delta^n)$, and thus it is also $RQ(T)$. Since $RQ(A)$ is the union over all nondegenerate simplices $T_i \to A$ of the subcomplexes $RQ(T_i) \to RQ(A)$, and since these subcomplexes have the same fiber products over $RQ(A)$ as over $RQ(\Delta^n)$, and since each $RQ(T_i)$, $RQ(\delta^n)$ is a monomorphism, $RQ(A) \to RQ(\Delta^n)$ is a monomorphism. This finishes the proof of (6.3).

LEMMA 6.5. *If $\Lambda \subset \Delta^n$ is the union of all but one face of dimension $n-1$, $Q(\Lambda) \to \Lambda \times \Lambda$ is an epicofibration.*

PROOF. Let $T_1, \ldots, T_n \subset \Lambda$ be the faces of Δ of codimension one. Each T_i is the cone, based on the vertex opposite the missing face, on some subcomplex of Λ. Thus, any subspace made from the T_i by intersections and unions will always be a trivial space, as it will be a cone.

Notice that $\Lambda \times \Lambda$ is the union of the subcomplexes $T_i \times T_j$, and that

$Q(\Lambda)$ is the union of the $T_i \times T_i$. We can construct $\Lambda \times \Lambda$ from $Q(\Lambda)$ by attaching the $T_i \times T_j$ for $i \neq j$ along appropriate subcomplexes. If we can show that $\Lambda \times \Lambda$ can be constructed from $Q(\Lambda)$ by attaching along epicofibrations, since the cobase extension of an epicofibration is again an epicofibration, we will see that $Q(\Lambda) \to \Lambda \times \Lambda$ is an epicofibration.

We now construct an increasing sequence $Q(\Lambda) = Y_0 \subset Y_1 \subset \cdots \subset Y_n = \Lambda \times \Lambda$ so that each $Y_i \to Y_{i+1}$ is an epicofibration.

We call a subcomplex $U \subset \Lambda$ a complete intersection if it is of the form $T_{i_1} \cap \cdots \cap T_{i_r}$. Since each T_i is the face opposite a vertex of the missing face, the complete intersections are the faces of Δ^n opposite the various faces of the missing face. Further, their representation as an intersection without repetition of the T_i is unique up to order. If U is a complete intersection, let U^* be the union of all the T_i which do not contain U, and denote by ∂U the intersection of U with U^*. Notice that every inclusion $\partial U \to U$ is a trivial cofibration.

We now filter $\Lambda \times \Lambda$ by an increasing filtration $\Lambda \times \Lambda = Y_{n-1} \supset \cdots \supset Y_0 = Q(\Lambda)$. To do this, let Y_t be the union of all $U \times V$ for U, V complete intersections of r and s terms respectively with $r + s = n + 1 - t$. Notice that if U and V have a term, say T_i, in common, $U \times V \subset T_i \times T_i \subset Q(\Lambda)$. Thus we may assume that Y_t is obtained from Y_{t-1} by attaching each $U \times V$ with no terms in common, and with ranks r, s respectively such that $r + s = n + 1 - t$. Now $(U \times V) \cap Y_{t-1}$ is just $((\partial U) \times V) \cup (U \times \partial V)$. Further, any pair $U \times V$, $U' \times V'$ as just described which are different, have their intersection inside Y_{t-1}. Thus Y_t is obtained from Y_{t-1} by the cobase extension of a coproduct of maps of the form $(\partial U \times V) \sqcup^{\partial U \times \partial V} U \times \partial V \to U \times V$, which are all epicofibrations. Thus $Q(\Lambda) \to \Lambda \times \Lambda$ is an epicofibration. This completes the proof of (6.5), and thus the proof of 6.2.

Suppose now that X, Y are fully fibrant simplicial sets. Then if $f\colon X \to Y$ is a fibration of simplicial sets, it will be an epifibration if $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a Kan fibration. To see this, observe that $\pi_0(f)$ is a Kan fibration if and only if for all trivial cofibrations $i\colon A \to B$ of finite simplicial sets, the map

$$\mathrm{Hom}(B, \pi_0(X)) \to \mathrm{Hom}(A, \pi_0(X)) \times_{\mathrm{Hom}(A, \pi_0(Y))} \mathrm{Hom}(B, \pi_0(Y))$$

is an epimorphism. However, since X is fully fibrant, $\mathrm{Hom}(B, \pi_0(X)) = \pi_0(\mathrm{Hom}(B, X))$, etc., so we see that f will be an epifibration precisely when $\pi_0(f)$ is a fibration of simplicial sets.

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