A CHARACTERIZATION OF HEREDITARY RINGS OF FINITE REPRESENTATION TYPE

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In this note we announce a characterization of hereditary rings R of finite representation type in terms of a sequence of partial Coxeter functors associated to R .

The methods we use are theory of almost split sequences [1], partial Coxeter functors [2], [3], [5], representations of species [7], and some ideas of Ringel [8, §6].

We recall from [7] that a *species* $M = (F_i, {}_iM_j)_{i,j \in I}$ is a finite set of division rings F_i and $F_i - F_j$ bimodules ${}_iM_j$. Throughout we will suppose M is a species such that ${}_iM_j \neq 0$ implies ${}_jM_i = 0$ for $i \neq j$. M is called *finite dimensional* if the dimensions

$$d_{ij} = \dim({}_iM_j)_{F_j}, \quad d_{ji} = \dim_{F_i}({}_iM_j)$$

are finite for $i \neq j$. We denote by $\mathcal{H}(M)$ the category of all finite dimensional right representations of M (see [6], [8]). From M we derive an *oriented valued graph* $(\Gamma, \mathbf{d}, \Omega)$ (not necessarily symmetrizable) with valued edges

$$\cdot \xrightarrow{(d_{ij}, d_{ji})} \cdot$$

$i \qquad \qquad j$

precisely when ${}_iM_j \neq 0$. Given source (resp. sink) k in $(\Gamma, \mathbf{d}, \Omega)$ we define a species $M^k = (F_i, {}_iN_j)_{i,j \in I}$ by taking

$${}_iN_j = \begin{cases} {}_kM_i^j = \text{Hom}_{F_i}({}_kM_i, F_i) & \text{for } j = k \quad (\text{resp. } = {}_jM_k^i \text{ for } i = k) \\ {}_iM_j & \text{for } j \neq k \quad (\text{resp. for } i \neq k). \end{cases}$$

If M is a finite dimensional species and k is a sink in $(\Gamma, \mathbf{d}, \Omega)$ we can define a pair of *Coxeter functors*

$$\mathcal{H}(M) \xrightleftharpoons[S_k^-]{S_k^+} \mathcal{H}(M^k)$$

having following properties:

(c₁⁺) S_k^- is left adjoint to S_k^+ .

(c₂⁺) Suppose $\mathbf{X} = (X_i, {}_j\rho_i)$ is indecomposable in $\mathcal{H}(M)$. Then

(i) $S_k^+\mathbf{X} = 0$ if and only if $\mathbf{X} \cong \mathbf{F}_k$, where $\mathbf{F}_k$ has F_k on k th coordinate and zeros otherwise,

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(ii) if $S_k^+ \mathbf{X} = 0$ then there is a natural isomorphism $S_k^- S_k^+ \mathbf{X} \simeq \mathbf{X}$ and $\dim S_k^+ \mathbf{X} = s_k(\dim \mathbf{X})$ where $\dim \mathbf{X} = (\dim(X_i)_{F_i})_{i \in I}$ and $s_k: Z^\Gamma \rightarrow Z^\Gamma$ is the corresponding reflection defined by using the matrix (d_{ij}) .

(c₃⁺) If $\mathbf{X}$ and $\mathbf{Y}$ are indecomposables in $\mathcal{H}(M)$ and $S_k^+ \mathbf{X} \neq 0$, $S_k^+ \mathbf{Y} \neq 0$ then there is a natural isomorphism $\text{Hom}(\mathbf{X}, \mathbf{Y}) \cong \text{Hom}(S_k^+ \mathbf{X}, S_k^+ \mathbf{Y})$.

(c₄⁺) If $0 \rightarrow \mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{Z} \rightarrow 0$ is an almost split sequence in $\mathcal{H}(M)$ with $\mathbf{X}$ noninjective and $S_k^+ \mathbf{X} \neq 0$ then $0 \rightarrow S_k^+ \mathbf{X} \rightarrow S_k^+ \mathbf{Y} \rightarrow S_k^+ \mathbf{Z} \rightarrow 0$ is almost split.

The properties (c₁⁺)–(c₄⁺) and analogous properties (c₁[−])–(c₄[−]) for the case k is a source can be found in [2], [5].

Now let M be a finite dimensional species, $(\Gamma, \mathbf{d}, \Omega)$ its valued graph and $k_1, \dots, k_n$ a fixed admissible orientation (see [5]). We define a sequence $\{k'_j\}_{j \in \mathbb{Z}}$, where $\mathbb{Z}$ is the set of integral numbers. We put

$$k'_j = \begin{cases} k_{r+1} & \text{if } j \geq 0, j = tn + r, 0 \leq r < n, \\ k_{n-r+1} & \text{if } j < 0, -j = tn + r, 0 \leq r < n. \end{cases}$$

Moreover for $m \in \mathbb{Z}$ we define inductively species $M^{(m)}$ putting $M^{(0)} = M$ and

$$M^{(m)} = \begin{cases} (M^{(m-1)})^{k'_{m-1}} & \text{for } m \geq 1, \\ (M^{(m+1)})^{k'_m} & \text{for } m \leq -1. \end{cases}$$

We say that M has *finite dimension property* if all species $M^{(m)}$, $m \in \mathbb{Z}$, are finite dimensional. In this case we denote by $(\Gamma, \mathbf{d}^m, \Omega^m)$ the valued graph of the species $M^{(m)}$ and by $s_m: Z^\Gamma \rightarrow Z^\Gamma$ the reflection $s_{k'_m}$ defined by using the matrix $\mathbf{d}^m = (d_{ij}^m)$. Following Ringel [8] we say that M has the *constant dimension property* if $d_{ij}^m = d_{ij}$ for all $i, j \in I$, $m \in \mathbb{Z}$.

Now given a species M with the finite dimension property we have an infinite diagram

$$\begin{array}{ccccccccc} \cdots & \longleftrightarrow & \mathcal{H}(M^{(-2)}) & \xrightleftharpoons[S_{-2}^-]{S_{-2}^+} & \mathcal{H}(M^{(-1)}) & \xrightleftharpoons[S_{-1}^-]{S_{-1}^+} & \mathcal{H}(M) & \xrightleftharpoons[S_1^-]{S_1^+} & \mathcal{H}(M^{(1)}) & \xrightleftharpoons[S_2^-]{S_2^+} & \mathcal{H}(M^{(2)}) & \longleftrightarrow & \cdots \\ & & \downarrow \text{dim} & & \downarrow \text{dim} & & \downarrow \text{dim} & & \downarrow \text{dim} & & \downarrow \text{dim} & & \\ \cdots & \longleftrightarrow & Z^\Gamma & \xrightleftharpoons[s_{-2}]{s_{-2}} & Z^\Gamma & \xrightleftharpoons[s_{-1}]{s_{-1}} & Z^\Gamma & \xrightleftharpoons[s_1]{s_1} & Z^\Gamma & \xrightleftharpoons[s_2]{s_2} & Z^\Gamma & \longleftrightarrow & \cdots \end{array}$$

where S_i^+ and S_i^- are appropriate partial Coxeter functors and s_i the corresponding reflections. The diagram will be called the *infinite Coxeter scheme* of M with respect to the admissible sequence $k_1, \dots, k_n$. Now we are able to formulate the main result of the paper.

THEOREM. *Let M be a finite dimensional species and suppose that its valued graph $(\Gamma, \mathbf{d}, \Omega)$ is connected. The category $\mathcal{H}(M)$ is of finite representation type if and only if M has the finite dimension property and there exists a*

number $m > 0$ such that $s_m \cdots s_1(e_i) \not\equiv 0$ for any source i of Γ , where $e_i \in Z^\Gamma$ is the vector having 1 on the i -th coordinate and zeros otherwise. Moreover, if m is minimal with the above property then

(a) The mapping $\dim: \mathcal{H}(M) \rightarrow Z^\Gamma$ is a one-one correspondence between isomorphism classes of indecomposable representations in $\mathcal{H}(M)$ and vectors in Z^Γ of the form $s_1 \cdots s_{t-1}(e_{k'_t})$ where $t < m$ and k'_t is a sink in $(\Gamma, \mathbf{d}^t, \Omega^t)$. In other words any indecomposable representation X in $\mathcal{H}(M)$ has the form $X \cong S_1^- \cdots S_{t-1}^- F_{k'_t}$, where $t < m$ and $F_{k'_t}$ is a simple projective in $\mathcal{H}(M^{(t-1)})$ with $F_{k'_t}$ on the k'_t th coordinate and zeros otherwise.

(b) There are equivalences $\mathcal{H}(M^{(i)}) \approx \mathcal{H}(M^{(m'q+i)})$ for $i = 0, 1, \dots, m' - 1$ and $q \in \mathbb{Z}$.

Observe that the theorem gives us a characterization of hereditary rings R of finite representation type having the property that R are generated over their centers by a set of cardinality $\aleph_c$, the first strongly inaccessible cardinal number. Indeed, if R is such a ring and M_R is the species of R then by [9, Corollary 4.6] there is an equivalence $\text{mod-}R \simeq \mathcal{H}(M_R)$.

Let us recall from [6] that if M is finite dimensional and the division rings $F_i, i \in I$, are finitely generated over their centers then M has the constant dimension property and therefore $\mathcal{H}(M)$ is of finite representation type if and only if its valued graph is a Dynkin diagram.

REMARK. *There exists a hereditary ring R of infinite representation type such that its valued graph $(\Gamma, \mathbf{d}, \Omega)$ is the Dynkin diagram*

$$\cdot \xrightarrow{(1, 2)} \cdot$$

(apply [2] and [4]).

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