reviewer and S. Saeki [**1983**]. Both of these latter papers were apparently of too recent origin to be included in this book. Actually, this book had apparently been in gestation for some time and was adumbrated by the paper of L. A. Rubel and B. A. Taylor [**1969**], which the reader can profitably consult for a short excursion into some of the ideas discussed here, as well as some variations on the proofs in the book.

In conclusion, I feel the authors completely achieved their goal and have presented their case in a very lively, concise manner. The density of errors is very low (but regrettably there is no index). The chapters are short, and each is followed by a number of relevant, accessible exercises. The book is rewarding reading for cognoscenti and students alike.

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*Spectral theory of Banach space operators*, by Shmuel Kantorovitz, Lecture Notes in Mathematics, Vol. 1012, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 179 pp., $9.50, 1984. ISBN 3-5401-2673-2

The earliest results of the spectral reduction theory for bounded and unbounded selfadjoint operators can be found in works by D. Hilbert, F. Riesz and J. von Neumann. The spectral theorem for bounded selfadjoint operators is essentially due to Hilbert, although he stated his results in terms of quadratic forms. The terminology which is in use today is closer to that of F. Riesz. Influenced by the development of quantum mechanics, von Neumann initiated the systematic study of unbounded selfadjoint operators. The interested reader can find other facts from this exciting history in the pertinent sections of the second volume of the treatise [**3**].

The spectral reduction theory for selfadjoint and normal operators has numerous applications in diverse fields of mathematics, such as topological groups, harmonic analysis, selfadjoint boundary value problems, almost periodic functions, etc. However, in some important problems there occur nonnormal operators for which, of course, the above-mentioned spectral reduction theory does not suffice. A precise formulation and a solution of the spectral reduction problem for bounded linear operators in Banach spaces are due to N. Dunford. Unlike the normal operators in Hilbert spaces, for which the existence of a resolution of the identity is a consequence of the normality, the so-called spectral operators, introduced by N. Dunford, are a priori associated with resolutions of the identity. A resolution of the identity is a homomorphism from the Boolean algebra of Borel sets in the complex plane into a Boolean algebra of projections on the given Banach space, which is countably additive in the strong operator topology. In addition, there exists an operator which commutes with the values of this homomorphism and satisfies some natural spectral inclusions. Such an operator is said to be spectral, and it determines uniquely a resolution of the identity. Details about spectral operators can be found in the third volume of [**3**].

There are many interesting operators, both bounded and unbounded, that have a resolution of the identity. And yet, by requiring the existence of an associated resolution of the identity, many elementary and often remarkable operators are ruled out. For instance, as U. Fixman has shown (see [**2**, p. 223]), even the usual shift operator on $l_p(-\infty, +\infty)$ fails to have a countably additive resolution of the identity unless $p = 2$. Another simple example of an operator that is not spectral is the multiplication operator with the independent variable on $C[\alpha, \beta]$. A more elaborate example of a differential operator that has no resolution of the identity is presented in [**5**]. But, as frequently happens in mathematics, more comprehensive concepts have been sought and eventually found. This time the main idea was to derive the spectral properties of an operator as consequences of an associated operational calculus rather than building the operational calculus from other spectral properties. The holomorphic functional calculus for arbitrary operators and the operational calculus with continuous (or bounded Borel) functions for a selfadjoint operator in a Hilbert space have been known for a long time. Therefore, it seemed natural to replace the algebra of bounded Borel functions by a smaller one, as done by some authors. However, it was C. Foiaş who systematically used the operational calculus as a tool in the study of spectral properties of operators [**4**] (see also [**3**, Part III] for other references). Later, I. Colojoară and C. Foiaş introduced a more general concept of an operational calculus by means of the so-called admissible algebras of functions [**1**].

From his earliest works, the author of the book under review has used the concept of an operational calculus as an approach to the spectral properties of operators. His results, published in various journals since 1964, are now brought together and presented in a unified and simplified manner. Although he works with operational calculi, his methods and objectives are essentially different from those in **[1]**. Let us briefly describe some topics dealt with in the present book.

Let $\mathbf{K}$ be a compact subset of the real line $\mathbf{R}$. Let $\mathscr{H}_{\mathbf{R}}(\mathbf{K})$ be the algebra of complex functions that are real analytic in a neighbourhood of $\mathbf{K}$. A topological algebra $\mathscr{A}(\mathbf{K})$ of complex functions defined in a neighbourhood of $\mathbf{K}$, with pointwise operations, such that $\mathscr{A}(\mathbf{K}) \supset \mathscr{H}_{\mathbf{R}}(\mathbf{K})$ topologically, is called a basic algebra. If $\mathscr{A}$ is a unital complex Banach algebra, an $\mathscr{A}(\mathbf{K})$-operational calculus for an element $a \in \mathscr{A}$ is a continuous representation $\tau: \mathscr{A}(\mathbf{K}) \to \mathscr{A}$ carried by $\mathbf{K}$ such that $\tau(t) = a$, where $t \to t$ is the identity in $\mathbf{K}$. In this case the element $a$ is said to be of class $\mathscr{A}(\mathbf{K})$. Under a certain homogeneity condition on a normable basic algebra $\mathscr{A}(\mathbf{K})$, the author proves that there exists an integer $n \geqslant 2$ such that the element $a$ as above is of class $C^n(\mathbf{K})$. This is what the author calls the "first reduction", showing that the abstract setting can be brought to a more concrete one.

Let $T_n$ be equal to $M + nJ$ on the space $C[\alpha, \beta]$ ($\alpha \leqslant 0 \leqslant \beta$), where $M$ is the multiplication operator with the independent variable and $J$ is the Volterra operator. This is an operator of class $C^n[\alpha, \beta]$, but not of class $C^{n-1}[\alpha, \beta]$. The "second reduction" theorem asserts that an element $a \in \mathscr{A}$ is of class $C^n[\alpha, \beta]$ if and only if there exists a continuous linear map $U$: $C[\alpha, \beta] \to \mathscr{A}$ normalized by the condition $U(1) = a^n/n!$, such that $L_a U = UT_n$, where $L_a(b) = ab$ for all $b \in \mathscr{A}$. Moreover, $U$ is uniquely determined and related to the $C^n[\alpha, \beta]$-operational calculus in a way similar to the Taylor formula. In other words, $T_n$ can be regarded as a universal model for elements of class $C^n$, providing a "weak representation" for them. This weak representation is the motivation of the author's effort to penetrate the intimate structure that turns $T_n$ into an operator of class $C^n$. On this line, the author proves the following general result. Let $S$, $V$ be two operators on a Banach space, satisfying what is designated as the "Standing Hypothesis" (in particular, $S$, $V$ satisfy the Volterra relation $[S, V] = V^2$). Set $T_\zeta = S + \zeta V$, where $\zeta$ is a complex number. If $S$ has real spectrum and is of class $C^m$, then $\sigma(T_\zeta) = \sigma(S)$ for all $\zeta$, and $T_\zeta$ is of class $C^{m+k}$ in the strip $|\mathrm{Re}\,\zeta| \leqslant k$. For $m = 0$ the condition $|\mathrm{Re}\,\zeta| \leqslant k$ is necessary and sufficient. Such classification results are then refined to the case $S + f(V)$, where $f$ belongs to a certain class of analytic functions.

Another type of problem is to determine the values of $\zeta$ such that $T_\zeta = S + \zeta V$ is spectral. For instance, if $S$, $V$ satisfy the Standing Hypothesis and $S$ is a spectral operator of scalar type (i.e., $S$ is the integral of the identical function with respect to its resolution of the identity) with real spectrum, then $T_\zeta$ is spectral if and only if $\mathrm{Re}\,\zeta = 0$. This result (combined with one mentioned above) illustrates the scarcity of the family of spectral operators in the family of operators of class $C^n$ ($n > 0$). (It will be further shown that, conversely, the "singular" $C^n$-operators are spectral.) The author then extends some of the above results to the case when $S$ is an unbounded operator. Assuming that $iS$

generates a strongly continuous group of operators, he can use methods from the theory of semigroups, instead of those of Banach algebras no longer available. Versions of the operational calculus and spectral decompositions, localized to some linear manifolds, for operators with real spectrum conclude this exposition.

The material is well written, the style is alert and attractive, despite the unavoidable technical portions. Many proofs are nice pieces of fine analysis. The author presents an original, interesting and consistent point of view concerning the spectral theory of linear operators, especially of those having real spectrum. The reviewer has several reasons to believe that the spectral theory of linear operators has much to gain from the systematic study of operators with "thin" spectrum, in particular of those with real spectrum. The present work is a remarkable illustration of this assertion.

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*The Cauchy problem*, by H. O. Fattorini, Encyclopedia of Mathematics and its Applications, Volume 18, Addison-Wesley, Reading, MA, 1983, xxii + 636 pp., $69.96. ISBN 0-201-13517-5

About three hundred years ago Isaac Newton taught us that the motion of a physical system is governed by an initial value problem or *Cauchy problem* for a differential equation, and the notion of Cauchy problem has been developing ever since. Here the phrase *differential equation* should be interpreted broadly so as to include systems of partial differential equations, integrodifferential equations, delay differential equations, and other kinds of equations. Most, but not all, of the Cauchy problems that arise "naturally" are *well-posed problems* —that is, problems for which a solution exists, is unique, and depends continuously on the ingredients of the problem. These requirements often necessitate imposing auxiliary conditions, such as boundary conditions, on a given Cauchy problem.

Of special interest are *linear* equations. There are two reasons for this. Firstly, many equations, such as the Schrödinger equation of nonrelativistic