When $k = \mathbf{Q}$ early workers appear to have believed that the rank (number of generators of infinite order) of the Mordell-Weil group is bounded; now the opposite is conjectured, but the truth is not known. The possible structures of the torsion part of the Mordell-Weil group have, however, been determined by Mazur using tools from the theory of modular forms.

Associated with an elliptic curve over a global field such as $\mathbf{Q}$ there is associated an L-function, many of whose properties remain conjectural. Guided first by a heuristic intuition and then by massive numerical computations, Birch and Swinnerton-Dyer were led to some very precise conjectures relating the behavior of the L-function to such things as the Mordell-Weil and Tate-Shafarevich groups. These conjectures have stood the test of much subsequent investigation, but it is only recently that some fragments of them have been proved.

The above partial account indicates the central position of the theory of elliptic curves and the wide variety of disciplines on which it draws. The author justifiably remarks "Considering the vast amount of research currently being done in this area, the paucity of introductory texts is somewhat surprising." In the reviewer's opinion his book fills the gap admirably. An old hand is hardly the best judge of a book of this nature, but the reports of graduate students are equally favorable.

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Geometry of CR-submanifolds, by Aurel Bejancu. Mathematics and Its Applications, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, and Tokyo, xii + 169 pp., $49.50. ISBN 90-277-2194-7.

The study of complex submanifolds of a Kaehlerian manifold, in particular, of a complex projective space, is one of the most important fields in differential geometry. It began as a separate area of study in the last century with the investigation of algebraic curves and algebraic surfaces in classical algebraic geometry. Included among the principal investigators are Riemann, Picard, Enriques, Castelnuovo, Severi, and C. Segre. It was J. A. Schouten, D. van Dantzig and E. Kähler [**5**, **8**, **9**] who first tried to study complex manifolds from the viewpoint of Riemannian geometry in the early 1930s. In their studies, a Hermitian space with the so-called symmetric unitary connection was introduced. A Hermitian space with such a connection is now known as a Kaehlerian manifold.

It was A. Weil [**10**] who in 1947 pointed out that there exists in a complex manifold a tensor field J of type $(1,1)$ whose square is equal to minus the identity transformation of the tangent bundle, that is, $J^2 = -I$. In the same year, C. Ehresmann introduced the notion of an almost complex manifold as an even-dimensional differentiable manifold which admits such a tensor field J of type $(1,1)$. The necessary and sufficient condition for an almost complex structure to be obtained from a complex structure was studied by Eckmann, Ehresmann and Frölicher in the case where the almost complex structure is of class C^ω and by Newlander and Nirenberg in the case where the almost complex structure is merely differentiable.

An almost complex manifold (respectively, a complex manifold) is called an almost Hermitian manifold (respectively, a Hermitian manifold) if it admits a Riemannian structure which is compatible with the almost complex structure J. The theory of almost Hermitian manifolds, Hermitian manifolds, and in particular Kaehlerian manifolds has become a very interesting and important branch of modern differential geometry. For recent results in this direction, see Kobayashi [**6**].

The study of complex submanifolds of a Kaehlerian manifold from the differential geometrical point of view (that is, with emphasis on the Riemannian metric) was initiated by E. Calabi [**2**, **3**] and others in the early 1950s. Since then it has become an active and fruitful field in modern differential geometry. Many important results on Kaehlerian submanifolds have been obtained, particularly in the last two decades. A nice survey article concerning differential geometry of Kaehlerian submanifolds was given by K. Ogiue [**7**] in 1974. Because many new results in this direction were proved after the appearance of Ogiue's article, an up-to-date and detailed monograph should be enormously helpful for readers interested in this direction.

Besides complex submanifolds of a Kaehlerian manifold, there is another important class of submanifolds called totally real submanifolds. A totally real submanifold of a Kaehlerian manifold M is a submanifold of M such that the almost complex structure J of M carries the tangent space of the submanifold at each point into its normal space. Totally real submanifolds and complex submanifolds of a complex projective space $\mathbf{CP}^n$ with the usual Fubini-Study metric are exactly those submanifolds of $\mathbf{CP}^n$ which are invariant with respect to the curvature transformation of $\mathbf{CP}^n$. In particular, this shows that every totally geodesic submanifold (or more generally, every parallel submanifold) of $\mathbf{CP}^n$ is either a totally real submanifold or a complex submanifold of $\mathbf{CP}^n$. Although complex submanifolds have been studied for a long time, the study of totally real submanifolds (in particular, from the differential geometric point of view) was initiated less than fifteen years ago. Since then, the theory of totally real submanifolds has undergone a rapid development.

The notions of complex submanifolds and totally real submanifolds were combined by the author in 1978 [**1**] to give the notion of CR-submanifolds. In fact, a submanifold N of a Kaehlerian manifold M with almost complex structure J is called a CR-submanifold of M if there exists a differentiable distribution D: $x \mapsto D_x$ on N such that

(a) D is a holomorphic distribution, that is, $JD_x = D_x$ for each x in N and

(b) the complementary orthogonal distribution $D^\perp$ of D is a totally real distribution, that is, $JD_x^\perp \subset T_x^\perp N$, for each x in N, where $T_x^\perp N$ is the normal space of N in M at x.

Totally real submanifolds, complex submanifolds, and real hypersurfaces of a Kaehlerian manifold are trivial examples of CR-submanifolds. CR-submanifolds of a Kaehlerian manifold have some fundamental properties. For instance, every CR-submanifold of a Kaehlerian manifold is a CR- (Cauchy-Riemann) manifold, the totally real distribution is completely integrable, the holomorphic distribution is a minimal distribution (which generalizes a well-known result concerning minimality of Kaehlerian submanifolds) and there is a canonical de Rham cohomology class associated with an arbitrary compact CR-submanifold of an arbitrary Kaehlerian manifold,.... Since 1978, the study of CR-submanifolds has become a very active subject in differential geometry. Many results have been obtained. The book under review introduces the readers to the main problems of the differential geometry of CR-submanifolds. The book includes all of the important results on CR-submanifolds obtained to date. This makes the book a uniquely comprehensive work in the field.

In Chapter I, the author presents some basic formulas, definitions, and theorems as the necessary background material.

Chapter II is devoted to the differential geometry of CR-submanifolds of almost Hermitian manifolds. The results of this chapter generalize some known results concerning CR-submanifolds of Kaehlerian manifolds.

Chapters III and IV are the core of the book. In Chapter III, the integrability theorems of the totally real distribution and the holomorphic distribution of a CR-submanifold of a Kaehlerian manifold are presented. Umbilical CR-submanifolds, normal CR-submanifolds, CR-products, and antiholomorphic submanifolds of a Kaehlerian manifold are studied in detail. Also the canonical de Rham cohomology class and its applications are presented.

Chapter IV deals with CR-submanifolds of a complex-space-form. By a complex-space-form we mean a Kaehlerian manifold with constant holomorphic sectional curvature. The author discusses Riemannian submersions in connection with CR-submanifolds, results on CR-products, mixed-foliate CR-submanifolds, generic submanifolds, and CR-submanifolds with semiflat normal connection in a complex-space-form.

In Chapter V, various generalizations of CR-submanifolds in a manifold with various structure are presented. For example, generic submanifolds and submanifolds of a Sasakian manifold or of a quaternionic Kaehlerian manifold analogous to CR-submanifolds are introduced and studied.

The interrelation of the geometry of CR-submanifolds with the general theory of CR-manifolds is studied in Chapter VI.

In the last chapter, the author presents the geometrical structures of space-time, Penrose's twistor space and physical interpretations of CR-structures.

In summary, the author did a very good job of arranging all of the important results on CR-submanifolds in one place. Since the original material is widely scattered in the literature, readers interested in CR-submanifolds will find it enormously helpful to have the results assembled in one place with a unified exposition. This book should be a valuable addition to all research libraries.

REFERENCES

1. A. Bejancu, *CR-submanifolds of a Kaehler manifold.* I, II, Proc. Amer. Math. Soc. **69** (1978), 135–142; Trans. Amer. Math. Soc. **250** (1979), 333–345.
2. E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. **58** (1953), 1–23.
3. ______, *Metric Riemann surfaces*, Ann. of Math. Studies (1953), 77–85.
4. B. Y. Chen, *CR-submanifolds of a Kaehler manifold.* I, II, J. Differential Geom. **16** (1981), 305–322; 493–509.
5. E. Kähler, *Über eine bemerkenswerte Hermitische Metrik*, Abh. Math. Sem. Univ. Hamburg **9** (1933), 173–186.
6. S. Kobayashi, *Recent results in complex differential geometry*, Jahresber. Deutsch. Math.-Verein. **83** (1981), 147–158.
7. K. Ogiue, *Differential geometry of Kaehler submanifolds*, Adv. in Math. **13** (1974), 73–114.
8. J. A. Schouten and D. van Dantzig, *Über unitäre Geometrie*, Math. Ann. **103** (1930), 319–346.
9. ______, *Über unitäre Geometrie konstanter Krümmung*, Proc. Kon. Nederl. Akad. Amsterdam **34** (1931), 1293–1314.
10. A. Weil, *Sur la théorie des formes différentielles attachées à une variété analytique complexe*, Comm. Math. Helv. **20** (1947), 110–116.
11. K. Yano and M. Kon, *CR-submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser, Cambridge, 1982.

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Noncommutative harmonic analysis, by Michael E. Taylor. Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, 1986, xvi + 328 pp., $68.00. ISBN 0-8218-1523-7

Harmonic analysis began as a technique for solving partial differential equations, in the work of Daniel Bernoulli on the vibrating string equation and Fourier on the heat equation. Since then, both subjects have blossomed into independent, wide-ranging, central mathematical disciplines with many subspecialties and with connections to almost all branches of mathematics, pure and applied. I do not think Bernoulli or Fourier would have been surprised by the developments in partial differential equations, but they surely would have been astounded by the growth of harmonic analysis. I also think they would have been pleased by Michael Taylor's new book, which explores some of the recent connections between harmonic analysis and partial differential equations, very much in the spirit of their pioneering work.

The modern definition of harmonic analysis is roughly the following: there is a linear space of functions $\mathscr{F}$, real or complex valued, ordinary or generalized, defined on a domain X on which a group G acts. One seeks first to identify those functions in $\mathscr{F}$ which transform in as simple a fashion as possible under G, then one seeks to expand the general function in $\mathscr{F}$ as a series or integral of these simple functions, and finally one seeks to use the expansion to solve problems which are compatible with the action of G. The simplest cases, such