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Theory of limit cycles, by Ye Yan-qian, Cai Sui-lin, Chen Lan-sun, Huang Ke-cheng, Luo Ding-jun, Ma Zhi-en, Wang Er-nian, Wang Ming-shu, and Yang Xin-an; translated by Chi Y. Lo. Translations of Mathematical Monographs, Vol. 66, American Mathematical Society, Providence, R. I., 1986, xi + 435 pp., $140.00. ISBN 0-8218-4518-7

The problem of the existence of limit cycles for certain nonlinear differential equations is one of the most fundamental areas of research in differential equations. This question has intrigued mathematicians since the discovery of Henri Poincaré [**4**] that certain nonlinear systems do in point of fact admit an oscillatory behavior. Indeed this work has motivated the creation of the fields of topological dynamics and general stability theory.

Part of the Hilbert sixteenth problem set forth at the 1900 International Congress of Mathematicians was concerned precisely with this topic. Explicitly, David Hilbert posed the problem of finding the maximum number of limit cycles for a first-order differential equation of the form

$$dy/dx = P(x, y)/Q(x, y), \tag{$*$}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials. To give the uninitiated reader an idea of the difficulty of this still unsolved problem, we should note that two decades passed after the problem was initially posed before the mathematician Henri Dulac [**2**] succeeded in proving that such an equation admitted only a finite number of limit cycles. (The book under review has an extensive list of references where the interested reader can find more details about the above works.) Even in the case in which $P(x, y)$ and $Q(x, y)$ are quadratic polynomials this problem still remains unsolved. The maximal number of limit cycles for such quadratic systems has been conjectured to be 3, 4, and 5 at various times. (See [**5**] and the references therein.)

From a more applied point of view, oscillatory phenonema have been discovered in great abundance in "nature", perhaps some of the most striking occurring in electrical engineering, and in particular in electronics. It was B. van der Pol [**6**] who first wrote down, in the mid-1920s, a nonlinear differential equation to describe the stable oscillations in the triode vacuum tube which had a tremendous impact on the field of electronics design and of course on applied mathematics. (It was actually the physicist A. Andronov [**1**] who proved that the closed isolated trajectory discovered by van der Pol was indeed a limit cycle in the sense of Poincaré.) It was van der Pol's discovery that provided a major impetus to the study of nonlinear ordinary differential equations in both engineering and mathematics, and in particular, to the nonlinear phenomenon of limit cycles. We should add that the van der Pol equation has become a standard topic in modern engineering electronics courses as well as in differential equations.

The book under review covers some of the basic material on the theory and properties of limit cycles, e.g., the Poincaré-Bendixson theorem, multiplicity and stability theory, structural stability, and some standard results on the existence and uniqueness of limit cycles. What I think makes the book unique is that it considers in great detail quadratic systems of ordinary differential equations, i.e., systems in which $P(x, y)$ and $Q(x, y)$ of equation $(*)$ above are quadratic polynomials. China has been a center of research on this important topic, and this book brings together for the first time many results previously scattered throughout the literature. Quite importantly, via certain normalizations the authors classify quadratic systems into three types, and much of the work is centered about discussing the various possibilities of the limit cycle structure in the different cases. I like very much the authors' ingenious use of the method of Dulac functions in deducing the nonexistence of limit cycles in certain circumstances. (This is probably a nice way of introducing a student to the Dulac theory in a very natural, concrete fashion.)

Of course, the subject of quadratic systems has been studied outside of China as well. For example, L. Markus [3] has an interesting paper on the global topological classification of the system $(*)$ when the polynomials are homogeneous. His point of view makes heavy use of the theory of nonassociative algebras. There has also been some nice work done in the Soviet Union on the subject. (See the extensive list of references in the book.)

I think it is essential to note that besides the purely mathematical importance of studying and understanding in detail the first nontrivial case of a nonlinear system of differential equations, the subject of quadratic systems is very interesting precisely because many nonlinear phenonema in science and engineering admit mathematical models which are quadratic. In particular some of the work covered in this book on quadratic systems has recently been found to be quite useful in aircraft control, especially in connection with the nonlinear feedback inversion techniques developed for the stabilization of modern aircraft. It turns out that almost always one cannot invert the complete dynamical system describing the behavior of the given aircraft, and in many cases the part not inverted (the "residual part" of the control system) has the form $(*)$ in which the $P(x, y)$ and $Q(x, y)$ are quadratic polynomials. The engineering problem then is that one must design a feedback compensator for the quadratic residual part of the system. Besides the verification that this residual part is stable (and what to do if is isn't!), a key question is the existence of limit cycles for the given admissible values of the system parameters. It is precisely such problems that this book considers in exquisite detail, and which therefore make it so useful.

In summary, I believe that the book under review is an important survey of a unique school of differential equations. Moreover, because the mathematicians involved have done such a beautiful job of studying the case of quadratic systems which has a number of very interesting physical applications, I am sure that this book should become very popular both in the mathematics and engineering communities.

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Vanishing theorems on complex manifolds, by Bernard Shiffman and Andrew John Sommese, Progress in Mathematics, vol. 56, Birkhäuser, Boston, Basel, Stuttgart, 1985, xiii + 170 pp., \$19.95. ISBN 0-8176-3288-3

On a topological or differentiable manifold local objects (functions, vector fields, differential forms, ...) can be extended to global ones by a partition of unity. On complex or algebraic manifolds this is generally not possible. The obstructions for doing this lie in a (first) cohomology group. Therefore it is of prime interest to know conditions guaranteeing the vanishing of such cohomology groups.

If the manifold is Stein or affine this is always the case but for compact complex manifolds the situation is completely different. In the compact case, however, the cohomology groups $H^q(X, F)$ are always finite dimensional $\mathbf{C}$-vector spaces (F being a coherent $\mathcal{O}_X$-module sheaf). Under the heading "vanishing theorems" we understand general statements when such groups do vanish, however. Here F mostly is a holomorphic line bundle or, more generally, a vector bundle of higher rank.

The most famous vanishing theorem is the one of Kodaira proved in 1953: *Let L be a positive holomorphic line bundle on the compact complex manifold X. Then*

$$H^q(X, L \otimes K_X) = 0 \quad \text{for } q \geqslant 1.$$

Here K_X denotes the canonical line bundle (i.e. K_X is the line bundle of holomorphic n-forms, $n = \dim_{\mathbf{C}} X$). The positivity of a line bundle can be defined in several ways:

(i) *differential-geometric*: L admits a hermitian metric h such that the curvature form $\theta_h = i\bar{\partial}\partial \log h$ is a positive $(1,1)$-form.

(ii) *function-theoretic*: the zero-section of the dual bundle L^{-1} admits a strongly pseudoconvex neighborhood.

(iii) *algebraic-geometric*: the sections of a high-power L^m embed X into a projective space.

(iv) *numerical*: $c_1(L \mid Y)^s > 0$ for all irreducible reduced analytic subspaces $Y \subset X$ of dimension s (here c_1 denotes the first Chern class).

The equivalence of these conditions is by no means obvious. For instance the equivalence of (i) and (iii) is the celebrated Kodaira embedding theorem and (iv) is the Nakai-Moishezon criterion.