3. L. Markus, *Quadratic differential equations and non-associative algebras*, Contributions to the Theory of Nonlinear Oscillations (S. Lefschetz et al., eds.), Ann. of Math. Studies, no. 45, Princeton Univ. Press, Princeton, New Jersey, 1960.

4. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*. I, II, J. Math. Pures Appl. **7** (1881), 375–422 and **8** (1882), 251–296.

5. Shi Song-ling, *A concrete example of the existence of four limit cycles for plane quadratic systems*, Sci. Sinica **23** (1980), 153–158.

6. B. van der Pol, *On relaxation-oscillations*, Philos. Magazine **2** (1926), 978–992.

ALLEN TANNENBAUM

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987

0273-0979/87 $1.00 + $.25 per page

Vanishing theorems on complex manifolds, by Bernard Shiffman and Andrew John Sommese, Progress in Mathematics, vol. 56, Birkhäuser, Boston, Basel, Stuttgart, 1985, xiii + 170 pp., $19.95. ISBN 0-8176-3288-3

On a topological or differentiable manifold local objects (functions, vector fields, differential forms,...) can be extended to global ones by a partition of unity. On complex or algebraic manifolds this is generally not possible. The obstructions for doing this lie in a (first) cohomology group. Therefore it is of prime interest to know conditions guaranteeing the vanishing of such cohomology groups.

If the manifold is Stein or affine this is always the case but for compact complex manifolds the situation is completely different. In the compact case, however, the cohomology groups $H^q(X, F)$ are always finite dimensional $\mathbf{C}$-vector spaces (F being a coherent $\mathcal{O}_X$-module sheaf). Under the heading "vanishing theorems" we understand general statements when such groups do vanish, however. Here F mostly is a holomorphic line bundle or, more generally, a vector bundle of higher rank.

The most famous vanishing theorem is the one of Kodaira proved in 1953: *Let L be a positive holomorphic line bundle on the compact complex manifold X. Then*

$$H^q(X, L \otimes K_X) = 0 \quad \text{for } q \geqslant 1.$$

Here K_X denotes the canonical line bundle (i.e. K_X is the line bundle of holomorphic n-forms, $n = \dim_{\mathbf{C}} X$). The positivity of a line bundle can be defined in several ways:

(i) *differential-geometric*: L admits a hermitian metric h such that the curvature form $\theta_h = i\bar{\partial}\partial \log h$ is a positive $(1,1)$-form.

(ii) *function-theoretic*: the zero-section of the dual bundle L^{-1} admits a strongly pseudoconvex neighborhood.

(iii) *algebraic-geometric*: the sections of a high-power L^m embed X into a projective space.

(iv) *numerical*: $c_1(L \mid Y)^s > 0$ for all irreducible reduced analytic subspaces $Y \subset X$ of dimension s (here c_1 denotes the first Chern class).

The equivalence of these conditions is by no means obvious. For instance the equivalence of (i) and (iii) is the celebrated Kodaira embedding theorem and (iv) is the Nakai-Moishezon criterion.

A consequence of the Kodaira vanishing theorem is the knowledge of

$$\dim_{\mathbf{C}} H^0(X, L \otimes K_X).$$

This follows from the Riemann-Roch-Hirzebruch theorem which says that $\sum_{q=0}^{n}(-1)^q \dim_{\mathbf{C}} H^q(X, L \otimes K_X)$ is a well-known expression depending on topological data (Chern classes of X and $c_1(L)$).

An important generalization of Kodaira's vanishing theorem is the one of Nakano,

$$H^q(X, \Omega_X^p \otimes L) = 0 \quad \text{for } p + q \geqslant n + 1.$$

Here $n = \dim X$ and Ω_X^p is the sheaf of holomorphic p-forms.

Kodaira and Spencer observed that the Kodaira-Nakano vanishing theorem implies the weak Lefschetz theorem with $\mathbf{C}$-coefficients:

$$H^q(X, Y; \mathbf{C}) = 0 \quad \text{for } q \leqslant n - 1,$$

where Y is a hyperplane section of the projective manifold X. Note that this is a topological condition for the complement $X \setminus Y$.

The method of proving the Kodaira-Nakano vanishing theorem is by complex differential geometry and analysis (the Laplace-Beltrami operator is elliptic). For a long time it was an open problem whether there is a purely algebraic proof of the Kodaira vanishing theorem. For surfaces this was shown to be true by Bogomolov. There are examples showing that the Kodaira vanishing theorem is wrong in positive characteristics. I do not know who was the first to observe that the weak Lefschetz theorem and the Hodge decomposition

$$H^k(X, \mathbf{C}) \simeq \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

imply the Kodaira-Nakano vanishing theorem. One reference I am aware of is an article of Faltings in Crelle's Journal in 1981. The Hodge decomposition is classically established by transcendental methods (elliptic theory). Recently a purely algebraic proof of the Hodge decomposition and therefore also of the Kodaira-Nakano vanishing theorem has been found. This heals an old wound of algebraic geometers.

For geometric applications—classification of submanifolds of projective space or birational classification of projective varieties—the Kodaira-Nakano vanishing theorem often is not sufficient. On one hand the line bundle L has to be replaced by a vector bundle of higher rank and on the other hand the positivity assumptions of the Kodaira-Nakano vanishing theorem have to be weakened.

For vector bundles of higher rank the positivity notions (i)–(iv) above do not coincide. Even in the differential geometric setting there are various different notions of positivity. The tangent bundle of $\mathbf{P}_n$, for instance, is positive in the sense of Griffiths but only semipositive in the sense of Nakano. I think the most important notion of positivity is the one of Grauert: This is the condition (ii) above. It is equivalent to "ampleness" of algebraic geometers.

The basic result for vector bundles of higher rank is the following isomorphism due to le Potier relating higher rank to the line bundle case:

$$H^q(X, \Omega_X^p \otimes E) \simeq H^q(\mathbf{P}(E), \Omega_{\mathbf{P}(E)}^p \otimes \mathcal{O}_{\mathbf{P}(E)}(1)).$$

Here $\mathbf{P}(E)$ denotes the projective bundle of hyperplanes in the fibres of E. The vector bundle E is Grauert-positive precisely if the line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ is positive. This implies immediately the vanishing theorem

$$H^q(X, \Omega_X^p \otimes E) = 0 \quad \text{for } p + q \geqslant n + r,$$

where $r = \operatorname{rank} E$, $n = \dim X$.

The normal bundle of a submanifold $X \subset \mathbf{P}_n$ is positive and application of vanishing theorems gives important information on possible X. The other generalization dictated by applications concerns the weakening of the positivity assumption of the line bundle L. If one assumes only that θ_h is semipositive and has everywhere at least $n - k$ positive eigenvalues then

$$H^q(X, L \otimes \Omega_X^p) = 0 \quad \text{for } p + q \geqslant n + k + 1.$$

A more important generalization is due to Ramanujam-Kawamata-Viehweg: Let L be a line bundle on the projective manifold X. Assume L to be *nef* and *big* (i.e. $c_1(L \mid C) \geqslant 0$ for all integral curves $C \subset X$ and $c_1(L)^n > 0$). Then

$$H^q(X, L \otimes K_X) = 0 \quad \text{for } q \geqslant 1.$$

Under these weakened assumptions the Kodaira-Nakano vanishing theorem is no longer true.

This vanishing theorem and variants of it play a crucial role in the birational classification of algebraic varieties. For instance, it can be shown that for projective manifolds X high powers of K_X are generated by global sections if K_X is nef and big. This implies in particular that

$$\bigoplus_{m \geqslant 0} H^0(X, K_X^m)$$

is a finitely generated $\mathbf{C}$-algebra. Therefore the canonical model

$$X_{\text{can}} = \operatorname{Proj}\Big(\bigoplus_{m \geqslant 0} H^0(X, K_X^m)\Big)$$

is well defined.

The book under review describes all this (with the exception of the last-mentioned application) and much more in only 160 pages. This is possible only through a very concise presentation of the material and in some cases the uninitiated reader has to work quite hard or to refer to other sources. The book seems to me very useful; it is suited for the advanced beginner but even experts will find some interesting novelties. It is surprising that the authors succeeded in presenting so much in so few pages.

I have, however, three points of criticism.

First, it is to be hoped that possible future editions will have an index.

Second, the book appeared at an unfortunate time because the last two years have been a very active period for vanishing theorems:

(1) Siu's solution of the Grauert-Riemenschneider conjecture.

(2) The approach of Esnault-Viehweg and Kollár towards vanishing theorems, which is related to the above-mentioned connection of weak Lefschetz and vanishing theorems.

(3) The "non-vanishing theorem" of Shokurov and the application of Kawamata et al. to $\bigoplus_{m \geqslant 0} H^q(X, K_X^m)$.

Third, the book deals only with the compact theory, the noncompact theory being mentioned only in the references.

In spite of these remarks, this book should have many readers since various mathematical fields come together in an exciting way: real analysis, complex differential geometry, complex analysis and algebraic geometry.

MICHAEL SCHNEIDER

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987

0273-0979/87 $1.00 + $.25 per page

Singularities of differentiable maps, Vol. 1. *The classification of critical sets, caustics and wave fronts*, by V. I. Arnol′d, S. M. Gusein-Zade, and A. N. Varchenko, translated from Russian by Ian Porteous. Birkhäuser, Boston, Basel and Stuttgart, 1985, x + 382 pp., $44.95. ISBN 0-8176-3187-9.

Singularity theory is a subject of comparatively recent origin, and spans a wide range of disciplines: one may study singularities of differentiable or of holomorphic maps, or of algebraic varieties, under differentiable or topological equivalences: thus topology, complex analysis, and algebraic geometry all play a part.

Although the theory has important global aspects, it is dominated by local considerations, and we will focus on these. The central object of study is the germ at a point of a C^∞-map between smooth manifolds. One commonly takes local coordinates in source and target: then the Taylor expansion of the map defines a k-jet (truncating the expansion at terms of degree k), and the germ is said to be k-determined (for the equivalence relation E) if all germs with the same k-jet are E-equivalent to it. The most important equivalence relations are defined by local diffeomorphisms of source or target or both; one may also wish to consider homeomorphisms. This idea, of approximating the infinite-dimensional space of germs by the finite-dimensional spaces of jets, is central to the whole subject.

A map is *stable* if all nearby (in the C^∞-topology) maps are equivalent to it. The equivalence may be by diffeomorphism of source and target, by homeomorphism, or many other choices. There are corresponding local notions; their precise definitions are rather technical.

Singularity theory has its origins in papers of Whitney and Thom: the latter full of ideas and proposals but not easy to follow at the time (1955–1965) when they were written. Towards 1970 several major developments gave this area of mathematics a new impetus and cohesion.