BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Malcev-admissible algebras, by Hyo Chul Myung. Progress in Mathematics, vol. 64, Birkhäuser, Boston, Basel, Stuttgart, 1986, xvi + 353 pp., $55.00. ISBN 0-8176-3345-6

In both art and science, the specialists in any subject find it necessary to establish a set of rules or formalisms to allow a good understanding of that subject to be developed. When a subject is well enough understood in terms of one set of rules, the deficiencies of that set of rules become apparent, and a new, frequently more general, set of rules or formalism is developed.

One of the areas where we are all aware of this process is in physics, where quantum theory followed Newtonian mechanics, and has in turn been added to and amended by a series of later theories. In this progression, increasingly sophisticated mathematics has been used, including much algebra. Lie groups and Lie algebras have become staples, and many other nonassociative algebras, including the octonions, have been used by different theoretical physicists. When Lie groups were first introduced into physics, they represented the symmetries or automorphisms of the physical systems that they were associated with, and the corresponding Lie algebras were used because they were easier to work with than the groups. Over a period of time, the relation between the physical system being investigated and the associated algebra has become more intimate. Today, if an algebra is associated with a physical system, and if the formalism allows a physicist to deduce useful information about the physical system from the associated algebra without completely solving the dynamical problems, then the algebra is fulfilling its role.

If one wants to construct a generalization of a physical theory, one natural way is to pick a generalization of the associated algebra, and to try to use appropriate formalism to derive a more general physical theory. It is just such an attempt which provides the motivation for the book *Malcev-admissible algebras* by Hyo Chul Myung. The physicist Ruggero Maria Santilli conceived the need to generalize the Lie algebras associated with certain physical theories to Lie-admissible algebras (definitions will be given shortly). In Santilli's words [**9**, p. 3], "Newtonian systems with forces not derivable from a potential can be represented with Hamilton's equations with external terms. As we shall see, the generalized brackets of the time evolution law induced by these broader equations violate the Lie algebra laws. But, when properly written, they characterize precisely a general Lie-admissible algebra. In particular, under certain restrictions, these generalized brackets characterize the subclass of flexible

Lie-admissible algebras." Again [9, p. 90], "The transition from contemporary physical models based on Lie algebras or their graded-supersymmetric extensions, to the generalized Lie-admissible models, essentially permits the treatment of particles as being extended, by therefore admitting additional, contact/nonpotential/non-Hamiltonian interactions." The interest of Santilli and certain other physicists in Lie-admissible algebras helped to stimulate further mathematical research on these algebras, and on Malcev-admissible algebras, which are a generalization. It is this research which is covered in Myung's book.

Let A be any algebra (associative or not) over a field F of characteristic not 2, and let A^- denote the algebra obtained from A by using the same addition and scalar multiplication by F, but by using the new product $[x,y] = xy - yx$ for $x, y \in A$, where juxtaposition indicates the original product in A. If A is associative, then the algebra A^- is automatically a Lie algebra (i.e., the identities $[x,y] = -[y,x]$ and $J(x,y,z) \equiv [[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ hold for all $x, y, z \in A^-$). We call an arbitrary (i.e., not necessarily associative) F-algebra A *Lie-admissible* if the associated algebra A^- is a Lie algebra. Dually, we can define the algebra A^+ by replacing the products in A by $x \circ y = (xy+yx)/2$, and the algebra A is called *Jordan admissible* if the algebra A^+ is a Jordan algebra (i.e., if $x \circ y = y \circ x$ and $(x \circ x) \circ (y \circ x) = ((x \circ x) \circ y) \circ x$). Any associative algebra is Jordan admissible. We call the algebra A *flexible* if $(xy)x = x(yx)$ for all $x, y \in A$.

These concepts were first defined by Albert [1], who also showed how Jordan admissible algebras can be analyzed using idempotents and Peirce decompositions. Lie-admissible algebras seemed less tractable, even after Laufer and Tomber [6] proved that if A^- is a simple Lie algebra, then $A = A^-$. The key to studying flexible Lie-admissible algebras was supplied by Anderson [2], who noted that, for a flexible algebra A, commutation by a fixed element is a derivation of A^+. Thus, if A is flexible and Lie-admissible, then A^+ is a Lie module for A^-, and hence so is A. Using this key, various results on the structure of flexible Lie algebras were established by Block, Benkart and Osborn, Myung and Okubo, and others.

Probably the most fascinating nonassociative algebra is the octonions $\mathcal{O}$, which can be thought of as having the basis $1, e_1, \ldots, e_7$ with multiplication given by

$$e_i e_{i+1} = e_{i+3} = -e_{i+1} e_i, \qquad e_{i+1} e_{i+3} = e_i = -e_{i+3} e_{i+1},$$
$$e_{i+3} e_i = e_{i+1} = -e_i e_{i+3}, \qquad e_i^2 = -1,$$

for $1 \leq i \leq 7$, and where subscripts are to be interpreted modulo 7. The algebra $\mathcal{O}$ is not Lie-admissible, but is very close in that $\mathcal{O}$ satisfies

$$(*) \qquad J(x,y,[x,z]) = [J(x,y,z),x]$$

rather than the relation $J(x,y,z) = 0$, which is satisfied in a Lie algebra. An algebra is called a *Malcev algebra* if it is anticommutative and satisfies $(*)$. Thus, $\mathcal{O}$ is Malcev-admissible. Myung has generalized many of the known results on Lie-admissible algebras to Malcev-admissible algebras, and his book gives a complete account starting from scratch of all of the known results on Lie-admissible and Malcev-admissible algebras. This book is an excellent source for anyone wishing to learn this theory. He also covers other topics in nonassociative algebras which have been of interest to physicists, such as power-associative products on matrices, and algebras whose derivations include sl(3) or G_2. The latter algebras have also been studied by many physicists who have not used Lie-admissible algebras. This book contains a lot of material, most of which has not appeared in any other mathematical book. However, the connections with physics are only referred to briefly. Those familiar with the area will note that many chapters of the book follow very closely the original papers on which they are based.

The biggest disappointment in the structure theory of flexible Lie-admissible algebras is that the plethora of simple algebras that exist appear to defy meaningful classification. Thus, the class of simple algebras that exist is much larger than the class of Lie algebras, and so the replacement by Santilli of Lie algebras by Lie-admissible algebras has vastly weakened one of the rules of the formalism. Santilli has faced this problem in practice by selecting for his work the Lie-admissible algebra A^*, which comes from an associative algebra A using the definition $x * y = xry - ysx$ for some fixed r, $s \in A$. These algebras are not flexible in general, and don't appear to have a natural intrinsic characterization. If the approach of using Lie-admissible algebras is to succeed in physics, it seems likely that some more natural mathematical or physical condition will have to be imposed on the Lie-admissible algebras that arise. The success of this approach will propel Myung's book into the limelight.

Independent of the fate of Lie-admissible algebras in physics, this book will probably serve for some time as the best source for those mathematicians who are interested in Lie-admissible algebras. The book is also recommended to physicists who are interested in constructing algebras with prescribed derivation algebras.

REFERENCES

1. A. A. Albert, *Power-associative rings*, Trans. Amer. Math. Soc. **64** (1948), 552–593.

2. C. T. Anderson, *A note on derivations of commutative algebras*, Proc. Amer. Math. Soc. **17** (1966), 1199–1202.

3. G. M. Benkart and J. M. Osborn, *Flexible Lie-admissible algebras*, J. Algebra **71** (1981), 11–31.

4. ——, *Flexible Lie-admissible algebras with the solvable radical of A^- abelian and Lie algebras with nondegenerate forms*, Hadronic J. **4** (1981), 274–326.

5. R. E. Block, *Determination of A^+ for the simple flexible algebras*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 394–397.

6. P. J. Laufer and M. L. Tomber, *Some Lie admissible algebras*, Canad. J. Math. **14** (1962), 287–292.

7. H. C. Myung, *Lie-admissible algebras*, Hadronic J. **1** (1978), 169–193.

8. S. Okubo and H. C. Myung, *Adjoint operators in Lie algebras and the classification of simple flexible Lie-admissible algebras*, Trans. Amer. Math. Soc. **264** (1981), 459–479.

9. R. M. Santilli, *Lie-admissible approach to the hadronic structure*, Vol. II, Hadronic Press, Nonantum, Mass., 1982.

J. MARSHALL OSBORN