and Bibliography of Riemannian Geometry, compiled by Bérard and Berger, with a partial update covering the period since 1982.

The book is not, and is not intended to be, a broad overview of the by now very large topics of direct and inverse problems in Riemannian geometry. It is, however, a clear account of the contributions along the above lines of the author and his collaborators, and some of its material is not in print elsewhere. Altogether, within the framework of its aims, the book conveys a clear account of this interesting work, and comprises, together with the recent book of Chavel [CH] on related topics, a very worthwhile addition to the literature of spectral geometry.

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Intégration et théorie des nombres, by Jean-Loup Mauclaire; preface by S. Iyanagi. Hermann, Paris, 1986, xi + 152 pp., 160F. ISBN 2-7056-6035-6

A function is arithmetic if it is defined on the positive integers. In this review arithmetic functions will be real or complex valued. The scope of this definition is rather wide, and functions of number theoretic interest generally have some structure attached to them. An example is the Dirichlet divisor function $d(n)$, which counts the number of distinct divisors of the integer n. Its values on the first ten integers are 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, and appear roughly increasing. Considered over the range $205 < n \leq 215$ however, we have 4, 6, 10, 4, 16, 2, 6, 4, 4, 4. It is characteristic of functions of number theoretic interest that their successive values sail so erratically about. I begin with a snapshot history of the methods devised in Analytic Number Theory to come to grips with this phenomenon. As in many a family album, some important relations do not get into the picture.

According to Dirichlet, it was Gauss who considered the mean-value

$$M(g,x) = x^{-1} \sum_{n \leq x} g(n) \tag{1}$$

of an arithmetic function g. Dirichlet, himself, employed this notion several times. He showed that

$$x^{-1}\sum_{n\le x} d(n) = \log x + (2\gamma - 1) + O(x^{-1/2}) \tag{2}$$

uniformly for $x \ge 1$, where γ is Euler's constant. A much more spectacular application appears in his celebrated proof that when the integers $a > 0$, b have no common factor other than 1, the arithmetic progression $am + b$, $m = 1, 2, \ldots$, contains infinitely many primes. Besides this, in his proof Dirichlet employed series of the form

$$G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}. \tag{3}$$

It is a mark of Dirichlet's consummate artistry in their application that they have been named after him.

The arithmetic function g is said to be *multiplicative* if it satisfies the relation $g(ab) = g(a)g(b)$ whenever the integers a, b are mutually prime. Since, apart from order, every positive integer can be represented uniquely as a product of primes p,

$$\sum_{n=1}^{\infty} g(n)n^{-s} = \prod_{p}(1 + g(p)p^{-s} + g(p^2)p^{-2s} + \cdots) \tag{4}$$

in the sense that if for some (real) s one side is absolutely convergent, then so is the other, and their values are equal. On the left is a Dirichlet series, on the right an Euler product. When $g(n)$ is identically 1 we get a further simplification

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p}(1 - p^{-s})^{-1}, \tag{5}$$

certainly for real $s > 1$. By letting $s \to 1+$ and noting that the series of reciprocals n^{-1} diverges, Euler could conclude that there were infinitely many primes. Dirichlet took this as his own starting point. For him the rôle of g was played by suitable extensions of the characters on an appropriate residue class group. The construction of these characters was another pioneering step along the way.

Ultimately Dirichlet needed information about the particular series (3) under consideration, as s approached 1. It was to this end that he employed Gauss' idea of a mean-value.

This approach to the properties of prime numbers was transformed by a suggestion of Riemann: Regard the function $\zeta(s)$ of the Dirichlet series at (5) as a function of the *complex* variable s, and examine its analyticity. From a knowledge of the zeros of $\zeta(s)$, and so of $-\log\zeta(s)$, we shall be able to deduce the distribution of the primes.

For the function $\zeta(s)$, now known as the Riemann zeta function, Riemann carried out a program of analytic continuation. In fact it is analytic over the whole plane save at $s = 1$ where, not unexpectedly, there is a pole. It satisfies a functional equation. There are no zeros of $\zeta(s)$ in the half-plane $\mathrm{Re}(s) > 1$, but infinitely many in the strip $0 \leq \mathrm{Re}(s) \leq 1$. Riemann's hypothesis is that these zeros all lie on the line $\mathrm{Re}(s) = 1/2$.

We should distinguish between two aspects of Riemann's view of Dirichlet series. The first is that if we know the sum function $G(s)$ well enough, then we can deduce the properties of the coefficients $g(n)$. This is exemplified in the multiplicative function $d(n)$. On the mth power of a prime p, $d(p^m) = m + 1$. An easy computation of the Euler product at (4) shows that the Dirichlet series corresponding to the divisor function is $(\varsigma(s))^2$. Using an integral transform and our knowledge of the behavior of $\varsigma(s)$, we can rederive Dirichlet's estimate for the mean-value of $d(n)$. Indeed, this method, embellished with methods for treating exponential sums, enabled the error term at (2) to be pushed down, so that by the early part of this century it went below $O(x^{-2/3})$. In this direction, the step from primes to integers, no information is required concerning the zeros of Dirichlet series.

The second aspect of the Riemann view is that if we know how to continue $G(s)$ analytically, and wish to derive the properties of g on the primes, we need the further information of when G vanishes. After Riemann's paper in 1859 it was nearly forty years before a method of any generality was found to guarantee a zero-free region. It required the existence of an Euler product, and when applied to $\varsigma(s)$ could show that there were no zeros on the line $\mathrm{Re}(s) = 1$. From this Hadamard and de la Vallée-Poussin in 1896 derived the first genuine asymptotic estimate

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1 \tag{6}$$

for the number $\pi(x)$ of primes up to a given magnitude x, and so realised part of Riemann's program. It has been a fight to improve the size of the zero-free region of $\varsigma(s)$, and despite brilliant and ingenious contributions, such as that of Vinogradov, we still do not know if there is a strip $\alpha < \mathrm{Re}(s) < 1$ with $1/2 < \alpha < 1$, in which $\varsigma(s)$ does not vanish. For the time being analytic number theory seems to be learning to live without the validity of the Riemann hypothesis. One might say that the step from integers to primes is deeper and more troublesome than the step from primes to integers.

Dirichlet was the successor of Gauss, at Göttingen, and Riemann the successor of Dirichlet. Their works, introduced in the mid-nineteenth century, gave a flying start to the study of arithmetic functions, introducing methods which remain lively to this day. Nonetheless, there are limitations to their approach. The multiplicative function defined by $g(p^m) = (-1)^{m+1}p^{m-1}$ has a sum function $G(s)$ defined by a series absolutely convergent in $\mathrm{Re}(s) > 1$. It has zeros at $(2k\pi i + \log(p+1))/\log p$ for integers k and primes p, and every point of the line $\mathrm{Re}(s) = 1$ is a limit point of these. There is no further analytic continuation of $G(s)$. Even for multiplicative functions we cannot generally expect to employ analytic continuation. Under the influence of the relation (5) we might retain the feeling of direction: primes to integers or, integers to primes, but general quantitative estimations would need a new aesthetic. In fact one was to come from the Theory of Probability.

Let f be a real-valued arithmetic function. For temporarily fixed $x \geq 1$, consider the frequency $F_x(z) = \nu_x(n; f(n) \leq z)$ amongst the integers n in the interval $1 \leq n \leq x$, of those for which the inequality $f(n) \leq z$ is satisfied.

Instead of considering the mean-values (1), we can investigate the limiting behavior, as $x \to \infty$, of the distribution functions $F_x(z)$. These distribution functions have the merit that unlike mean-values they are not particularly sensitive to isolated large values of f, however they are less convenient to work with.

A real function f is deemed *additive* if for mutually prime integers a, b it satisfies the relation $f(ab) = f(a) + f(b)$. The choice $f(q) = 1$ for all prime-powers q defines $\omega(n)$, which counts the number of distinct prime divisors of the integer n. In 1917 Hardy and Ramanujan proved that if $\varepsilon > 0$, then in a well-defined sense

$$|\omega(n) - \log\log n| < (\log\log n)^{1/2+\varepsilon}$$

holds for almost all integers n. Their proof went by induction on the integral values of $\omega(n)$, and was rather special in nature. They asked what other arithmetic functions essentially increased in this manner. Seventeen years went by and then a second proof of their result was given by Turán. As he showed, the interesting feature of Turán's proof method was that it would apply to an arbitrary additive function. In a form arrived at by Kubilius in 1962, it could be made to yield

$$(7) \qquad \sum_{n\leq x}\left|f(n) - \sum_{q\leq x} q^{-1}f(q)\right|^2 \leq c_1 x \sum_{q\leq x} q^{-1}|f(q)|^2, \qquad x \geq 1,$$

with absolute constant c_1 and prime-powers q. As Turán wrote to me in 1976, there was not the slightest sign that anyone suspected the existence of such general inequalities.

Beginning in the thirties of this century, a number of authors studied arithmetic functions implicitly in terms of the behavior of the frequencies $F_x(z)$. By 1938 Erdős, who made wide use of the method of Turán, explicitly proved that for a real additive arithmetic function the convergence of the three series

$$(8) \qquad \sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)|\leq 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\leq 1} \frac{f(p)^2}{p}$$

was sufficient to ensure that the frequencies $\nu_x(n; f(n) \leq z)$ converge weakly to a limiting distribution as $x \to \infty$. This result may also be applied to certain positive-valued multiplicative functions g, for then $\log g(n)$ is additive.

From a present day perspective, the Turán-Kubilius inequality looks remarkably like Tchebyshev's inequality for uncorrelated random variables, and the condition of Erdős' theorem like Kolmogorov's three-series criterion for the almost-sure convergence of a series of independent random variables. In fact neither Turán nor Erdős knew much of probability, as they themselves told me. The foundations of the theory of probability had been given acceptable clarity with the axioms of Kolmogorov in 1933, the year before Turán's paper appeared.

A method of probability, rather than the application of an aesthetic, was introduced into the study of arithmetic functions by Kac, who viewed the divisibility of integers by differing primes in terms of the independence of random variables. It was only approximate independence, but it enabled the

Central Limit Theorem to be applied. Thus in 1939 Erdős and Kac proved that

$$\nu_x(n; \omega(n) - \log\log x \le z\sqrt{\log\log x}) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du, \qquad x \to \infty.$$

The contribution of Erdős to this joint work was to overcome the approximate nature of the independence by the application of the (complicated) method of Brun's sieve. Note that in the spirit of the theory of probability, the arithmetic function $\omega(n)$ is renormalised in terms of the unbounded moving parameter x.

The method of Erdős and Kac ensured a similar convergence to the normal law for a wide class (though not all) of additive arithmetic functions. In particular, Kac showed that

$$\nu_x(n; d(n) \le 2^{\log\log x + z\sqrt{\log\log x}}) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du, \qquad x \to \infty.$$

From a probability point of view, Hardy and Ramanujan would have said "normally", the divisor function $d(n)$ is about $(\log n)^{\log 2}$ in size, with $\log 2 = .6931\ldots$ A comparison with Dirichlet's estimate (2) shows that the mean value of $d(n)$ is forced up by relatively infrequent large values.

In a joint work with Wintner, Erdős proved that the conditions (8) are also necessary for the weak convergence of the frequencies $\nu_x(n; f(n) \le z)$. To this end they applied the results of Erdős and Kac. In the light of the history of the prime number theorem, it is not surprising that this step, 'from integers to primes,' required more effort.

The work of Erdős and Kac was clarified and extended by Kubilius, who in 1954/55 constructed a finite probability space on which to model the behavior of additive arithmetic functions by sums of independent random variables. Altogether this approach is successful only if the function involved is not too often large on the prime-powers. This restriction reflects not only the nature of the model, but also the limitations of the underlying sieve method. In particular, the method fails if $f(q) = (\log q)^\alpha$ for a fixed $\alpha > 1$.

There is a method in the theory of probability, due to Lyapunov, which does not initially involve independence. The weak convergence of the frequencies $\nu_x(n; f(n) \le z)$ is equivalent to the uniform convergence, on compact t-sets, of the characteristic functions

$$\int_{-\infty}^{\infty} e^{itz}\, d\nu_x(n; f(n) \le z), \qquad t \text{ real}.$$

A straightforward calculation shows that this characteristic function is

$$[x]^{-1} \sum_{n \le x} g(n),$$

with a multiplicative function $g(n) = \exp(itf(n))$ that satisfies $|g(n)| = 1$ for all positive integers n. We are back to mean values!

In view of this equivalence, and our earlier remarks concerning the unlikelihood of analytically continuing the corresponding Dirichlet series $G(s)$, the situation in 1960 might not have seemed hopeful. Indeed, the Möbius function $\mu(n)$, which is $(-1)^{\omega(n)}$ when n has no square divisor greater than 1, and is zero otherwise, is readily checked to be multiplicative. From the existence of the mean-value

$$A = \lim_{x \to \infty} x^{-1} \sum_{n \le x} \mu(n)$$

it is easy to deduce that $A = 0$ and then, as was shown by Landau in 1911, the basic prime number theorem (6). Any further progress in the study of multiplicative arithmetic functions would apparently involve a proof of the prime number theorem along the way. And so it turned out, although the events were in a different order than expected.

In 1949, Erdős and Selberg found an elementary proof of the prime number theorem, one that did not apply functions of a complex variable. This was quite against the intuition of Hardy who, in a Copenhagen lecture of 1921, asserted that the existence of such a proof would cause a severe reappraisal of the subject. Hardy died just two years before the elementary proof was given.

The basis of this elementary proof was Selberg's formula

$$\sum_{p \le x} (\log p)^2 + \sum_{pr \le x} \log p \log r = 2x \log x + O(x),$$

where p, r denote primes, and which could be obtained using elementary manipulation. Even today it is not obvious that this will lead to a proof of the prime number theorem, and the subsequent elementary proofs vary widely in structure.

Although pushed far in the study of prime numbers, these elementary methods do not presently yield as much as will the classical method based upon the theory of the Riemann zeta function. However, the original proof of Erdős and Selberg certainly changed the view of the subject, since one might hope to apply similar methods to the study of arithmetic functions in some generality. To this extent, Hardy was correct.

In 1961, Delange proved that for complex-valued multiplicative functions g which satisfied $|g(n)| \le 1$ for all n, the mean values (1) could converge to a limit $A \ne 0$, as $x \to \infty$, if and only if the series $\sum p^{-1}(1-g(p))$ converged, and $g(2^r)$ was not -1 for at least one positive integer r. The existence of a nonzero mean value meant that as $s \to 1$, a tauberian theorem could be applied to the Dirichlet series $\log G(s)\zeta(s)^{-1}$. Since the limit A was not allowed to be zero, this result did not have the prime number theorem as a corollary, but it gave a new proof of the Erdős-Wintner theorem, through the method of characteristic functions.

This same year Wirsing began his researches in the asymptotic behavior of mean-values of multiplicative functions, culminating in a method, employing convolutions of functions and approximate integral equations, that enabled him to consider convergence of the frequencies (1) to zero, provided the values of g, $|g(n)| \le 1$, do not essentially fill up the whole unit disc $|z| \le 1$ in the complex z-plane. In particular, he proved that a limiting mean-value always existed for real multiplicative functions which satisfy $|g(n)| \le 1$. In this way he recovered the prime number theorem, with a proof different from, but philosophically similar to, that of Erdős and Selberg.

Looking more deeply into the structure of multiplicative functions g, Wirsing conjectured that if $|g(n)| \leq 1$ for all n, then as $x \to \infty$,

$$\sum_{n\leq x} g(n) = AxL(\log x) + o(x) \tag{9}$$

for some slowly oscillating function $L(u)$, $|L(u)| \equiv 1$, and constant A. This result was only available to him with the above restrictions upon the values of g, and he felt that an appropriate analytic method should apply to the study of such problems.

Presently such an analytic method was found by Halász. It employs the Dirichlet series $G(s)$, but the argument is carried out in the half-plane $\text{Re}(s) > 1$ of absolute convergence—we creep asymptotically close to the suspected pole at $s = 1$. The factorisation $G' = (G'/G)\cdot G$ is an essential hinge of the method. When g is multiplicative, $|g(n)| \leq 1$, the Euler product representation of $G(s)$ ensures that the Dirichlet series expansion of G'/G has manageably small coefficients.

Broadly speaking, although this does not do it justice, the method of Halász allows recovery of the mean-values (1) by contour integration from the appropriate behavior of the Dirichlet series $G(s)$. Thus for multiplicative functions g with $|g(n)| \leq 1$, $M(g,x) \to A$ as $x \to \infty$ if and only if $G(s) = A(s-1)^{-1} + o((\text{Re}(s)-1)^{-1})$ as $\text{Re}(s) \to 1+$, uniformly on each half-strip $|\text{Im}(s)| \leq M$. For such multiplicative functions the possible behavior(s) of $G(s)$ as $s \to 1$ could be classified, and Halász established Wirsing's conjecture. Once again we get a proof of the prime number theorem, but the elementary method of Erdős and Selberg, having served its purpose, is now a long way back.

These results of Delange, Wirsing, and Halász are of a general nature. When a multiplicative function g is not appreciably greater than 1 in absolute value, and its behavior on the primes is sufficiently known, a nontrivial estimate for its mean-value can be deduced. However, many interesting and difficult problems remain. I give a few in illustration.

If an additive function $f_x(n)$ depends upon x, what condition should it satisfy on the prime numbers in order that the frequencies

$$\nu_x(n; f_x(n) \leq z) \tag{10}$$

converge weakly as $x \to \infty$? The characteristic function of this frequency differs negligibly from $M(g,x)$ with the multiplicative function given by $g(n) = \exp(itf_x(n))$. To apply the method of Halász the corresponding Dirichlet series $G(s)$ is considered on the line $\text{Re}(s) = 1 + (\log x)^{-1}$, so that now the coefficients of $G(s)$ depend upon s. What should we expect for the asymptotic behavior of $G(s)$ as $s \to 1$?

If we restrict our attention to a given additive function f, and in the spirit of the Erdős-Kac theorem consider its renormalisations, we might try to obtain the conditions necessary for the weak convergence of the corresponding frequencies

$$\nu_x(n; f(n) - \alpha(x) \leq z\beta(x)) \tag{11}$$

heuristically, by continuing to apply the notion of Erdős and Kac that an additive function may be regarded as a sum of independent random variables.

This leads to wrong answers. In particular, for unbounded $\beta(x)$ it would guarantee all the limit laws of the frequencies (11) to be infinitely divisible; and it is known that they are not. The function defined by $f(q) = (\log q)^\alpha$, with any positive $\alpha \neq 1$, provides a counterexample.

At the American Mathematical Society meeting in St. Louis, in 1972, I made the following suggestion. Let us regard the Turán-Kubilius inequality as a bound for an operator norm, and employ the correspondence: If *operator to sufficiency*, then *dual operator to necessity.* It is often the case in projective geometry, where the duality is between point and line, that the dual of a proposition is its converse. This offered the possibility of a uniform approach to the study of the distribution of additive and multiplicative arithmetic functions. As illustrative examples I gave a new proof of the Erdős-Wintner theorem, and a new result concerning the strong law of large numbers for additive functions.

Applied to the study of the frequencies (11) this point of view enabled me to show that for convergence to the improper law with a $\beta(x)$ that does not increase too rapidly, $\alpha(x)$ must satisfy an approximate functional equation. The appropriate Dirichlet series $G(s)$ then satisfies an approximate differential equation. As a consequence it effectively has a simple pole, not at 1 but at a point near to 1, whose position varies with x (and so s).

These results enabled me to characterize those additive functions which have a nondecreasing normal order in the sense of Hardy and Ramanujan. Since Birch had shown that the only multiplicative functions with this property were the powers of n, their original question was answered for the two widest classes of arithmetic functions. Afterwards, Ruzsa gave a characterization of those frequencies (10) which converge to the improper law, using a sieve, probability theory, and the method of Halász.

For convergence of the frequencies (11) to proper laws, satisfactory necessary and sufficient conditions I could obtain only under the assumption of a growth condition upon the renormalising parameter $\beta(x)$. Whilst growth conditions of this type may be necessarily satisfied, we cannot presently prove so. Necessary and sufficient conditions for the weak convergence of the frequencies (10) to a proper law are currently not known.

Another illustrative problem is appropriate to the book under review, and I give specific references. In the early seventies I was interested in obtaining a generalisation of the result of Delange that would allow the multiplicative function g under consideration to sometimes assume values greater than 1. My aim was a set of conditions concerning the behavior of g on the integers, including the existence of a mean-value, which would in their entirety be equivalent to a set of conditions on the prime-powers. By analogy with the Lebesgue classes on the reals, it was natural to introduce for each $\alpha > 0$ the class L^α of arithmetic functions h for which

$$||h||_\alpha = \limsup_{x\to\infty} \left(x^{-1} \sum_{n\le x} |h(n)|^\alpha \right)^{1/\alpha}$$

was finite. Membership of a class L^α would ensure that a multiplicative function g was usually not too large. If the mean-value were nonzero, then $g(n)$ could not be too often small, either. This would allow the methods of probabilistic number theory to come into play, and so give reasonable control over the values of g on the primes p. However, to effect a complete result it would be essential to deduce from the bound on $M(|g|^\alpha, x)$ control over the large values of g on the prime-powers, and the result of Delange was here of no help in the formulation of an appropriate condition.

I considered first the case $\alpha = 2$. In view of the above remarks concerning the failure of exact independence when applied to divisibility, I applied the dual of the Turán-Kubilius inequality for $\alpha = 2$,

$$\sum_{q \le x} q \left| \sum_{n \le x, q \| n} a_n - \frac{1}{q} \sum_{n \le x} a_n \right|^2 \ll x \sum_{n \le x} |a_n|^2,$$

with $a_n = g(n)$, and obtained the necessary control [**8**]. In a subsequent paper [**9**] I extended this result. In order that a multiplicative function g belong to a class L^α with $\alpha > 1$, and have a nonzero limiting mean-value, it is necessary and sufficient that the series

$$\sum \frac{g(p) - 1}{p}, \qquad \sum_{1/2 \le |g(p)| \le 3/2} \frac{|g(p) - 1|^2}{p},$$

$$\sum_{||g(p)|-1| > 1/2} \frac{|g(p)|^\alpha}{p}, \qquad \sum_{p, m \ge 2} \frac{|g(p^m)|^\alpha}{p^m} \tag{12}$$

converge, and for each prime p,

$$\sum_{m=1}^{\infty} \frac{g(p^m)}{p^m} \ne -1.$$

Moreover, omitting the final one, these conditions guarantee that $|g|^\alpha$ has a finite limiting mean-value. To this end I introduced high-power versions of the Turán-Kubilius inequality [**11**]. I found it convenient to employ the method of Halász as well as other methods from probabilistic number theory.

The machinery of these papers was sufficient to characterize those multiplicative functions g which belonged to a class L^α and satisfied $||g||_\beta > 0$ for some $\beta < \alpha$. As a consequence, for multiplicative functions in L^α, those with a limiting mean-value zero could also be characterized in terms of their behavior on the primes [**9**, Theorem 2]. (Note that in the statement of condition (iv) of that theorem, $g(p)$ should read $|g(p)|$.) A natural generalisation of the representation (9) was obtained.

An independent generalisation of the case $\alpha = 2$ to $\alpha > 1$ for nonzero mean values was given by Daboussi [**2**]. His method, like that of Delange, employed the Dirichlet series $G(s)$ as $s \to 1$ through real values, so avoiding contour integration. A form of tauberian theorem is implicit in the argument.

In this account of the theory of arithmetic functions I have emphasized results of wide generality, rather than of particular depth. Even so, there are glaring omissions, such as sieve methods, and the work of Weyl on uniform distribution. In particular, I have not mentioned automorphic functions. This is a subject which blooms now so large that pages would not suffice, so I will settle for an example.

Ramanujan's function $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{j=1}^{\infty}(1-x^j)^{24}.$$

Ramanujan conjectured, and in 1917 Mordell proved with the aid of modular functions, that τ is multiplicative. From that moment modular functions found increasing application to number theory. Ramanujan made another conjecture: that $|\tau(p)| < 2p^{11/2}$. This lay much deeper.

In 1934 Rankin [**26**] proved that the function $\tau(n)^2 n^{-11}$ has a nonzero mean-value. The main ingredient of his proof was an analytic continuation of the corresponding Dirichlet series past the line $\mathrm{Re}(s) = 1$, so that the more classical machinery of analytic number theory could be applied to it. Here the use of modular functions was essential, in a sense representing a far development of a method of Riemann. As a corollary, a nontrivial estimate could be obtained for $\tau(n)$. Ultimately, the proof of Ramanujan's second conjecture was achieved by Deligne, when he established the analogue of the Riemann hypothesis for the local zeta functions in algebraic geometry. Apparently Deligne was influenced by Rankin's paper.

More recently, analytic continuation of specific Dirichlet series has been obtained by employing the resolvent of an appropriate relative of the Laplace operator.

What can a general consideration of multiplicative functions say about $\tau(n)$? As a result of my work on functions of class L^α, $\alpha > 1$, I could prove that if a nonnegative multiplicative function g possessed a mean-value, then so did its powers g^δ, $0 < \delta < 1$. Moreover, all of these new mean-values would be zero, unless the series $\sum p^{-1}(\sqrt{g(p)} - 1)^2$, taken over the prime numbers, converged. Applied to Ramanujan's function this showed that the limits

$$A_\delta = \lim_{x\to\infty} x^{-1} \sum_{n\le x} \left(\frac{|\tau(n)|}{n^{11/2}}\right)^\delta$$

existed for $0 < \delta < 2$. Since the side condition involving $\sqrt{g(p)}$ would have contradicted a well-known conjecture of Sato and Tate, I conjectured that the limits A_δ were zero [**11**].

That this is indeed the case was shown by Elliott, Moreno, and Shahidi [**16**], who proved that if $g(n) = |\tau(n)|n^{-11/2}$, and $0 < \delta < 2$, then $M(g^\delta, x) \ll (\log x)^{-\gamma}$ for some positive constant γ. The value of this constant was improved by Rankin [**27**], who implicitly demonstrated the validity of my related conjecture that no limiting mean-value A_δ with $\delta > 2$ exists. These results employ the analytic continuation of $\sum_{n=1}^{\infty} \tau(n)^4 n^{-22-s}$, with its double pole at $s = 1$, as obtained by Shahidi [**30**], and Moreno and Shahidi [**23**]. The general theory of multiplicative functions here served in a modest, motivational rôle.

It was conjectured by Lehmer that τ never vanishes. According to the Sato-Tate conjecture, the angles defined by $2\,\mathrm{Cos}\,\theta = \tau(p)p^{-11/2}$ have asymptotic density $2(\mathrm{Sin}\,\theta)^2/\pi$ over the range $0 < \theta < \pi$.

To give a final perspective on the estimation of arithmetic functions, I note that there is a positive constant μ, whose value can be calculated [**12**], so that almost certainly

$$\nu_x\left(n; \frac{|\tau(n)|}{n^{11/2}} \leq \frac{e^{z\mu\sqrt{\log\log x}}}{\sqrt{\log x}}\right) \Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt,$$

as $x \to \infty$. Even assuming the above conjectures, and employing the result of Deligne, this appears presently beyond reach.

I turn now to the book under review. There are four chapters: 1: Almost periodic sequences, 2: Generalised multiplicative functions and spaces of integrable functions, 3: Arithmetic mean and generalised multiplicative functions, 4: Applications of the results of the preceding chapters to various problems of number theory.

In Chapter 3 the author shows that if for $\alpha \geq 1$ the conditions (12) are satisfied by a multiplicative function g, then both g and $|g|^\alpha$ have nonzero limiting mean-values. This is carried out not for functions defined on the integers, but rather for functions on semigroups. These semigroups are to be freely generated by countably many generators, and to possess a norm N given by a semigroup homomorphism into the reals > 1. In addition they must satisfy two distributional properties.

H(L): $$\lim_{x\to\infty} x^{-1} \sum_{Nn\leq x} 1 \textit{ exists and is nonzero},$$

H(C): $$\sum_{Np\leq x} \log Np \ll x.$$

Here I have employed p to denote a typical generator, and n to denote a typical element of the semigroup. The integral ideals in the ring of algebraic integers of a finite extension of the rationals form such a semigroup. The method might be viewed as elementary, save that it appeals to results established in Chapter 2.

Chapter 2 is implicitly concerned with showing that for a multiplicative function in L^α to have a nonzero limiting mean-value, a condition equivalent to (12) is also necessary.

For a fixed prime p one may define a probability measure on the set $S_p = \{p^{-n},\ n \geq 0\}$ by $\mu_p(\{p^{-n}\}) = p^{-n}(1-p^{-1})$. The direct product of the S_p, together with the product measure μ induced on it by these marginal measures, then gives a format within which the divisibility of an integer by a particular prime may be viewed as a measurable event. The behavior of an additive function $f(n)$ over the interval $1 \leq n \leq x$ may be simulated by a sum of random variables, independent with respect to μ, provided that f can be replaced by a suitably truncated form f_r which vanishes on the prime powers $q > r$. Without application of a sieve method, this truncation must be severe, say down to $r < \log x$. The limiting behavior of the frequencies $\nu_x(n; f(n) \leq z)$ is then replaced by that of the $\nu_x(n; f_r(n) \leq z)$.

This is the method which implicitly underlined Erdős' work on arithmetic functions in the early part of this century, save that it was in a different format. For him the measure μ was replaced by asymptotic density, and the independence with respect to μ by application of the Chinese Remainder

Theorem in arithmetic. The truncation was down to a value of r independent of x.

This approach, attractive in its immediacy, has been rediscovered many times. Experience shows that it is useful only if the frequencies $F_x(z)$ under consideration have the property that if they are compact, then suitably translated they are also convergent. In particular, in this form it cannot be applied to the study of arithmetic functions with unbounded renormalisations, such as $(\omega(n) - \log\log x)/\sqrt{\log\log x}$.

An interesting variant was suggested by Schwarz and Spilker [**29**]. Give S_p the discrete topology, and then a one-point compactification. Assign the extra point measure zero. The product S of the compactified S_p then yields a compact space, with measure μ, in terms of which we can formulate Riesz representations.

In Chapter 2 the author follows the philosophy of Erdős within the construction of Schwarz and Spilker. The proof employs Dirichlet series $\sum g(n)n^{-s}$ partly after the manner of Delange [**5**], and Daboussi [**2**]. Unfortunately the setting is even more general than that of Chapter 3. Each S_p is replaced by an arbitrary denumerable set of elements $\{t_p^{(n)},\ n \geq 0\}$, and the primes p are then replaced by the positive integers as an index. The resulting (compactified) product T is given what is essentially a norm N, most easily viewed in terms of the semigroup norm considered in Chapter 3. A product measure on T is induced by the marginal measures μ_m with

$$\mu_m(\{t_m^{(n)}\}) = (1 - N(t_m^{(1)})^{-1})N(t_m^1)^{-m}.$$

The Dirichlet series is replaced by a formal (Euler) product.

The novelty of the author's approach is to show that the existence of a nonzero limiting mean-value for a multiplicative function is, under suitable conditions, equivalent to the existence of a related function in $L^1(T, \mu)$ whose integral over the whole space does not vanish. An attractive idea which is buried in the generality of the setting. It is also disappointing that amongst the many pages of details it is still necessary to appeal to Kolmogorov's three-series theorem from the theory of probability proper.

In spite of what the author says (cf. p. 70), I do not feel that this chapter clarifies "what a multiplicative function is", or what problem was "actually treated" in related considerations by other authors.

Chapter 1 contains background material and preparatory results for the following three chapters. It begins with the Bohr compactification of the group of integers, and goes on to discuss various spaces on which to study generalisations to arithmetic of Bohr's notion of an almost periodic function. The following examples are much studied.

For $\alpha \geq 1$, an arithmetic function f is said to be *almost periodic* B^α (B conveniently denotes both Bohr and Besicovitch) if for each $\varepsilon > 0$ there is a trigonometric function

$$P_\varepsilon(n) = \sum_{j=1}^{k} c_j e^{2\pi i \alpha_j n}, \qquad \alpha_j \text{ real},$$

so that $||f - P_\varepsilon||_\alpha < \varepsilon$. Clearly such functions form a subspace of L^α. It is straightforward to show that for real α the Fourier-Bohr coefficients

$$\hat{f}(\alpha) = \lim_{x \to \infty} x^{-1} \sum_{n \leq x} f(n) e^{-2\pi i \alpha n}$$

exist. Those α for which $\hat{f}(\alpha) \neq 0$ define the spectrum of f. In particular, the limiting mean-value $\hat{f}(0)$ exists. Apart from its intrinsic interest, the study of mean-values may be brought to bear upon that of almost periodic arithmetic functions.

A function is *limit-periodic* B^α if it is almost periodic B^α with a spectrum containing only rational points.

In Chapter 1, as in the rest of the book, the author studies the relevant Dirichlet series only in the neighborhood of a point on the real line, essentially on the real line. There is no discussion of analytic continuation, nor of Fourier inversion in the complex plane.

Chapter 4 specialises to functions defined on the integers. Employing results from the previous chapters, the author characterises multiplicative functions which are limit-periodic B with nonempty spectrum (this last condition is omitted from the formulation) in terms of their behavior on the primes, and the behavior of associated Dirichlet series. He goes on to show that a multiplicative function in B is limit-periodic B.

A characterisation of the multiplicative functions in B^α ($\alpha > 1$) with a nonempty spectrum was first given by Daboussi [**1**], using the L^2 Turán-Kubilius dual. As a result he showed that multiplicative functions belonging to B^α are automatically limit-periodic B^α. In fact more is true (see Daboussi and Delange [**3**]).

As a next application in Chapter 4, the Erdős-Wintner theorem for additive functions f is derived, by introducing the multiplicative function $g(n) = \exp(itf(n))$, as did Delange. Further results concerning associated limit-periodic sequences are established, following in part the early procedure of Erdős. The author's comments on page 139 concerning the inevitability of computing the mean value of g seem contradicted by this early work of Erdős.

The last result in the book concerns Ramanujan's function $\tau(n)$. Let $h(n) = \tau(n)n^{-11/2}$. It is proved that the map on the space $C(S, \mathbf{R})$ defined by

$$f \mapsto \lim_{\sigma \to 1+} \zeta(\sigma)^{-1} \sum_{n=1}^{\infty} \frac{f(n)h(n)^2}{n^\sigma} = \langle f, h^2 \rangle$$

has a Riesz representation $\int f \, d\nu$ over S, where the measure ν is singular with respect to the canonical measure μ introduced earlier in this review.

Underlying the method of this monograph is a limitation that is carefully skirted. The method will not deal with arithmetic functions, such as $|\tau(n)|n^{-11/2}$, which have asymptotic mean-value zero.

At whom is this book aimed? As an introduction to the theory of multiplicative and other arithmetic functions it would not serve. The rather large technical background assumed of the reader is out of proportion with the number and nature of the results proved. At rock bottom many of the proofs rely on manipulation of a number-theoretic nature, which might have been presented with less sophistication. Thus on p. 115 of Chapter 3, it is asserted that the simplest method of reducing the cases $\alpha > 1$ of a theorem to the case $\alpha = 1$ is a detour through the measure-theoretic results of Chapter 2. It seems simpler and more direct, to me, to note that the inequality $|z| - z \ll |z-1|^2$ is valid in the complex disc $|z-1| \leq 1/4$, and then to check the appropriate inequalities directly.

There is no general index, and no list of symbols. The set $\mathbb{N}^*$, of positive integers, is defined in the body of the text at the foot of page 4. This symbol occurs many times. However, in Chapter 2, §2 during the definition of the foundational products on which the measure theory is to be constructed comes, like a bolt from the blue, a set $\mathbb{N}$ as well. Moreover, we are told, and it is underlined in the text, that it is the choice of (norm) given by $N(t_u^{(n)}) = N(t_u^1)^n$ which simplifies the exposition. Examination of the subsequent text shows that $\mathbb{N}$ must be the set of nonnegative integers. Actually, it has occurred very much earlier, undefined, a couple of times, the first being Theorem 2.c on page 11 of Chapter 1. There, in one paragraph we find the nightmare notation: $\{g_k\}$, $k \in \mathbb{N}$, followed in the next sentence by $\{F_x(N), N \in \mathbb{N}^*\}$. The text appears to have been reduced directly from a typescript, and the sad fact is that the difference between N and $\mathbb{N}$ pretty well disappears.

On p. 70 of Chapter 2 we learn that the norms $N(t_u^{(1)})$ are assumed nondecreasing as u runs through $\mathbb{N}^*$. It is unfortunate that in the review of the earlier models which begins Chapter 3, this condition is omitted. From the detailed argument of p. 112, it is clear that some uniform bounding of $N(t_u^{(1)})$ from below is assumed, so presumably this (essential) monotonicity is actually still in force. Look out if you start with Chapter 3!

The content is rather uneven in level of sophistication. On page 38 there is a laborious summation by parts given in full detail. This could have been avoided by an application of the inequality $p_r \gg r \log r$ which follows immediately from the Tchebyshev inequality given three lines earlier. Whereas, on page 147 it is asserted of $h(n) = \tau(n)n^{-11/2}$, where $\tau(n)$ is Ramanujan's function, that: $h(p)^2 < 4$ is a result of Deligne, and $h(p^\alpha)^2 \leq (\alpha+1)^2$ is a simple consequence; no mention of a reference, nor of the theory of modular functions!! There are other 'references by name only'.

In spite of his efforts—and the peculiar comments which he gives as a gloss for some of the results do not help—one gains a rather fragmentary feel for the subject, with not much motivation for the proofs. For the beginner it could not help to be told (p. 115): "One considers the expression: ..." followed by an immediate dive into a large number of consecutive pages of crabbed symbology.

For the expert it is a different matter. It is interesting and useful to have standard theorems put into a different setting, and much fun can be had spotting familiar ideas in an unfamiliar guise. It is unfortunate that in his aim at the widest generality, the author has obscured the potential elegance of his approach.

I would have liked direct references to related works, rather than the selective and sometimes oblique references to only studies involving arithmetic functions with nonzero means. Examples would be Wirsing [**32**] and Halász [**18**], whose papers in the theory of arithmetic functions I regard as essential reading. These are nowhere mentioned. It would also have been appropriate to indicate Beurling as a founder of the abstract modelling of the multiplicative properties of the rational integers.

A comment is in order here, concerning the author's attitude towards the prime number theorem and the Turán-Kubilius inequality, whose rôle in the probabilistic theory of numbers he dismisses almost pejoratively in the introduction. A particular bête noire seems to be the Turán-Kubilius inequality. This he regards as apparently alien to the number theory considered, despite its rôle in the discovery of the results, and he emphasizes that it never appears in his present work (pp. 130, 145, respectively). The coup de grâce is apparently given on p. 145, where he indicates that his methodology allows extension of the results of §2 of his last chapter to more general semigroups, and for this reason the use of the Turán-Kubilius inequality (other results are hinted at) has been studiously avoided. The implication is that the Turán-Kubilius inequality is not then applicable. This is false.

Appropriate versions of the Turán-Kubilius inequality, valid for every $\alpha > 1$, may be readily derived in all the situations which he considers, both the semigroups, and the products of discrete spaces. Moreover, it does not need the full hypothesis H(L) of his treatment, merely the weaker bound $O(x)$ for the number of elements n with $Nn \leq x$. Of course, one cannot derive such analogues using the classical method of Turán (and Kubilius), but it is possible to apply another method, depending upon a simple application of Cauchy's theorem in the theory of complex variables, which is exposed in my papers [**11**, **15**] and book [**13**]. This is not surprising. Since the study of a group and its dual is tantamount to a consideration of the same object from two sides, the existence of any direct sum or product structure practically guarantees the existence of an inequality of Turán-Kubilius type.

The author's treatment of a particular selection of topics in terms of integration on a suitable space is interesting and valuable. To gain much from it, however, it would be better to bring to it an already well-defined grip on some basic theorems.

In sum, this is not a book with which to begin the study of arithmetic functions. Experts might well find it stimulating. It made me jump up and down a few times.

Notes. Dirichlet's remarks concerning mean-values may be found in his collected works [**6**]. An interesting account of his life is given in [**24**].

Classical accounts of Analytic Number Theory may be found in the books of Landau, e.g. [**22**]. There is a recent new edition of the well-known volume by Titchmarsh [**31**], edited by Heath-Brown. See also Prachar [**25**], Edwards [**7**], Ellison and Mendès-France [**17**], Davenport [**4**], and Ivić [**18**].

For a detailed account of probabilistic number theory up to 1964, including the work of Erdős, Wintner, Kac, and himself, see Kubilius [**21**]. For later developments, including Mean-Value theorems, see Elliott [**10**], and the supplement to [**13**]. There are a number of variant proofs of the mean-value theorem for multiplicative functions of class L^2, due to Delange and Daboussi, Schwarz and Spilker, and others. A survey of these, along with related references, may be found in Schwarz [**28**].

An introductory discussion of the estimates involving Ramanujan's τ-function, including the work of Rankin, may be found in Hardy [**19**].

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Le problème des modules pour les branches planes, by Oscar Zariski, with an appendix by Bernard Teissier. Hermann, Paris, 1986, 209 pp., $28.00. ISBN 2-7065-6036-4

In the fall of 1973 Oscar Zariski gave a series of lectures about curve singularities at the École Polytechnique in Paris. A set of notes based on these lectures was prepared by François Kmety and Michel Merle, and an appendix was added by Bernard Teissier. These notes have now been published as a book by Hermann.

Nothing of comparable originality has been published about the subject since the work of Enriques and Chisini [**5**]. The book describes a deep and beautiful analogy between the moduli space $\mathcal{M}_g$ for smooth curves of genus g, and a certain local moduli space $\mathcal{M}_\Gamma$ for plane curve singularities. A partial description of $\mathcal{M}_\Gamma$ is given, and many important open problems are described.

Riemann noticed that smooth curves of genus g depend on $3g-3$ parameters if $g > 1$. How many parameters are needed to describe plane curve singularities of the same topological type? It is remarkable that we are still unable to solve this problem in general, and in this book the reader will find the first real progress toward a solution.

Chapter I, "Préliminaires", Chapter II, "Invariants d'équisingularité", and Chapter III, "Représentations paramétriques", give a clear review of the classical theory of plane curve singularities. Chapter IV, "L'espace des modules", and Chapter V, "Étude des exemples..." give detailed calculations. Chapter VI, "Le point de vue de la théorie des déformations", and Teissier's appendix, describe the modern theory in which deformation theory plays a central role.

The vanishing of a polynomial $f(X,Y) \in \mathbf{C}[X,Y]$ defines an affine plane curve. A singularity of this curve, for example at the origin, is described as follows. As an element of the power series ring $\mathbf{C}[[X,Y]]$, $f(X,Y)$ will factor into a finite number of irreducible power series, with multiplicities. An irreducible factor $g(X,Y)$ defines a *branch* C of the singularity, with coordinate ring $\mathcal{O} = \mathbf{C}[[X,Y]]/(g) = \mathbf{C}[[t^n, y(t)]]$. In other words the problem of studying the singularity can be reduced to the problem of studying complete local