for arbitrary r. It is worth remarking here that Gromov's original paper on this subject [G3] only gave full details of the proof in the 1-dimensional case, leaving a large part of the proof for arbitrary n up to reader's imagination. This generalization looked plausible, but it wasn't at all clear how one might avoid a horribly messy argument. Gromov has now expressed his ideas much more fully, and has completed the proof in a very elegant, if abstract, way.

To end with, here are some comments on the book as a whole. It has been carefully written. The main theorems are clearly stated and their proofs, insofar as I have studied them, are accurate and quite detailed. The book is also essentially self-contained, so that it should be accessible to anyone who has a knowledge of the basics of differential topology and geometry. But one also needs a good deal of persistence, since it is easy to be overwhelmed by the wealth of new ideas and the many, very varied examples which accompany each theorem. And so one must make a considerable initial effort to understand the basic ideas and language and to learn one's way around. However, it's well worth it. The book is a wonderful treasure house of ideas.

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DUSA MCDUFF

Fundamental principles of the theory of extremal problems, by Vladimir M. Tikhomirov; translated by Bernd Luderer. John Wiley and Sons, Chichester, New York, Brisbane, Toronto and Singapore, 1986, 136 pp., $27.00. ISBN 0-471-905631

This translation (not always felicitous) from the original German version of 1982 is essentially a reprise of [IT], which appeared first in 1974. That book showed, efficiently and attractively, how nonlinear functional analysis in conjunction with the convexity methods of Fenchel, Moreau, and Rockafellar could supply a unified treatment for problems of variational calculus and optimal control. Unlike its predecessor (whose 450-page English translation is unfortunately out of print), this brief monograph is not self-contained, and

draws repeatedly on [**IT** and **ATF**], among others, especially for proofs of existence of solutions to extremal problems. Moreover, sufficient conditions are barely mentioned. What remains is the author's distillation, based on some twenty years' experience, of the principles available for obtaining necessary conditions for the solutions to (smooth) extremal problems. These principles, set forth in successive chapter headings, are those of Lagrange for constrained problems, duality in convex analysis and convex programming, extension in variational problems, and complete constraint removal.

Extremal problems have been associated with mathematics since antiquity and their formulation and resolution by variational methods are among the first successful applications of the calculus. Efforts since 1940 to extend the classical framework to embrace problems arising in the optimal control of systems have generated renewed interest in the subject.

In a modern setting, the theory of extremal problems is concerned with examining the points $\hat{x}$ which could supply a minimum (or maximum) value to a real-valued function f on a (Banach) space X, when restricted to a set $A \subseteq X$. Assuming that f is, say, Fréchet differentiable at $\hat{x} \in A^\circ$, then the Fermat Principle implies the familiar necessary condition for minimization, viz, that the Fréchet derivative

$$(1) \qquad\qquad f'(\hat{x}) = 0.$$

Moreover, when f is convex on X, then

$$(2) \qquad f(x) - f(\hat{x}) \geq f'(\hat{x})(x - \hat{x}), \qquad \forall x \in X,$$

and (1) suffices to infer that $\hat{x}$ is a global minimizer for the problem, even when $\hat{x}$ is a boundary point of A. However, in the latter case, (1) is no longer necessary for minimization. Instead, as Rockafellar has shown (in [**Ro**]), when A is convex and closed we should expect the inclusion

$$(3) \qquad 0 \in \partial(f + \delta A)(\hat{x}) = \partial f(\hat{x}) + \partial(\delta A)(\hat{x}),$$

where $\partial f(\hat{x}) = \{x^* \in X^* : f(x) - f(\hat{x}) \geq x^*(x - \hat{x}), \forall x\}$ defines the subdifferential of f at $\hat{x}$, and δA is the convex indicator function of A with values $+1$ on A, $+\infty$ otherwise. It is also known (e.g., from [**ET**]) that a minimizing $\hat{x}$ exists if, in addition, f is lower semicontinuous on a reflexive X and coercive on A (in that $\|x\| \to +\infty$ on $A \Rightarrow f(x) \to +\infty$), since then each sublevel set $\{x \in A : f(x) \leq c\}$ is (weakly) compact.

We may regard (3) as an expression of the Lagrange Principle, which asserts that constrained extremal problems can be attacked by considering a suitable related problem without constraints. To obtain a form which has more familiar classical antecedents, suppose that $A = \{x : F(x) = \mathcal{O}\}$, where $F : X \to Y$ is a Banach space mapping assumed continuously Fréchet differentiable at $\hat{x}$ with derivative $F'(\hat{x}) : X \to Y$ linear and continuous. Then if $F'(\hat{x})$ is onto (or has a range of finite codimension), there exists $\lambda \in \mathbf{R}$ and a $y^* \in Y^*$ for which the Lagrangian

$$(4) \qquad \mathcal{L} = \lambda f + y^* \circ F \text{ satisfies (1), i.e, } \mathcal{L}'(\hat{x}) = 0.$$

This result is a consequence of Lyusternik's work (from 1934) in characterizing tangency for sets as smooth as A. λ and y^* may be termed Lagrange multipliers, and under certain additional conditions it can be shown that $\lambda \neq 0$.

When $Y = \mathbf{R}^n$, inequality constraints in the form $F(x) \leq 0$ (interpreted componentwise) can also be handled, and there result the Kuhn-Tucker conditions of linear programming (from 1951), namely that (4) holds with λ and each component of $y^* \in \mathbf{R}^n$ nonnegative, while each summand in the dot product $y^* \cdot F(\hat{x})$ vanishes. Conversely, it is trivial to see that when $\lambda \neq 0$, these conditions are sufficient for the constrained minimization provided that $\hat{x}$ minimizes $\mathcal{L}$ (and this would be assured by the convexity of $\mathcal{L}$). The only new material in the present book is the discussion in Chapter 4 of recent work by Clarke, Ioffe, Miljutin, and others to the effect that if $\hat{x}$ minimizes f when restricted to the null set of F, as above, then with sufficient differentiability and regularity, there exists a $y^* \in Y^*$ for which $\hat{x}$ minimizes locally the penalty-like function $f + y^* \cdot F + \varepsilon\|F(\cdot)\|$, for each $\varepsilon > 0$. This result embodies the principle of complete constraint removal, which is regarded as a culmination of Lagrange's prescription for treating extremal problems with side conditions (discussed thoroughly in Chapter 1), and it is used to obtain second-order conditions necessary for minimization.

The author uses these results to obtain the standard necessary conditions for (smooth) problems in the classical variational calculus where, say

$$(5) \qquad\qquad f(x) = \int_a^b L(t, x(t), \dot{x}(t))\, dt,$$

on an appropriate (Sobolev) space of functions on $[a, b]$, and the constraints can be described by suitable functions F.

Much of the book is directed toward yet another exploration of convexity and its dominant role in both variational calculus and optimal control. (Among similar recent expositions are [**BP**, **Ce**, **Cl**, **ET**, **Sm**, **Tr** and **Ze**].) For the minimization of integral f as above, the convexity of L in $\dot{x}$ is known to be of vital importance for necessity (Weierstrass, c. 1879), sufficiency (Weierstrass, 1879; Hilbert, 1900, field theory) and existence (Tonelli, 1915). (Surprisingly unexplored until recently is the fact that convexity of L in $\dot{x}$ *and* x produces a convex f which affords elementary sufficiency arguments; see [**Ew** and **Tr**]). In particular, Tonelli showed that such convexity was responsible for the requisite lower semicontinuity of f. (In [1], Cesari provides a lively account of the contributions of Tonelli and his successors.)

Convexity in u of $L(t, x, u)$ is of equal significance to problems in optimal control in which it is desired to minimize a performance integral $f(x, u) = \int_a^b L(t, x, u)\, dt$ when the state $x(t) \in \mathbf{R}^n$ is determined from the "control" $u(t) \in U \subseteq \mathbf{R}^k$ through a given system of differential equations $\dot{x} = g(t, x, u)$. In most applications, the controls are discontinuous, and U is not open, so that classical variational methods cannot be used to characterize optimality in u. Instead, from 1961, we have the Pontrjagin conditions asserting that the values of an optimal control $\hat{u}$ should minimize at a.e. $t \in [a, b]$ those of an associated integrand $h(t, \hat{x}(t), u) = \lambda L(t, \hat{x}(t), u) + p(t) \cdot g(t, \hat{x}(t), u)$ for all $u \in U$, where $\lambda \in \mathbf{R}$, p is an appropriate adjoint or costate function, and $\hat{x}$ is the associated optimal state. In simple cases, when enough convexity is present, this result can be motivated by sufficiency considerations (as in the [**Tr**] supplement) and Tikhomirov shows how it can be derived (in Chapter 1) from an amalgam of earlier results, tapping the appropriate convexity which resides in integral

functionals as revealed by Lyapunov's Theorem. When L is not suitably convex, one remedy may be found through the Extension Principle in the form of Bogoljubov's Convexification Theorem (of 1930), which in Chapter 3 is related to chattering in control theory and given a relatively accessible proof, wherein the principal ingredient is Fenchel (second) conjugation. (The Fenchel conjugate (or polar) of $f\colon X \to (-\infty, +\infty]$ is the convex function f^* on the dual X^* defined by

$$f^*(z^*) = \sup_{x \in X} [z^*(x) - f(x)].$$

Its conjugate f^{**}, similarly defined, provides explicitly the greatest convex minorant of f.)

However, there is little attempt to utilize this same duality (developed in Chapter 2) to explore the Hamiltonian formulation of the problems considered (as in [Ro]), nor is there any presentation of the extensions of the results to nonsmooth extremal problems (as in [Cl]). Since these newer tools have been used recently by Clarke, Vinter and others [2, 3, 4], to shed further light on fundamental properties of solutions to classical (smooth) problems, it seems unfortunate that Tikhomirov has not employed his evident expositional talent to incorporate them in this latest version of his work.

Within its restrictions the book fulfills most of the goals stated by its author in a preface notable for its candor. Moreover, it is rich in historical-philosophical commentary and its length recommends it as an adjunct to a graduate level course in this subject. However, with few worked examples, and no problem sets, it probably would not serve as the text.

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<div align="right">JOHN L. TROUTMAN</div>

Quadratic forms and Hecke operators, by Anatolij N. Andrianov. Grundlehren der mathematischen Wissenschaften, vol. 286, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xii + 374 pp., $102.00. ISBN 3-540-15294-6

This book is an important contribution to analytic number theory, especially to the branch which is collected in *Mathematical Reviews* under the title "Modular Theory". Before we come to our subject we have to tell something of the background in which Hecke's theory assumes a conspicuous place. In this we allow ourselves some simplifications.

A question which has attracted mathematicians since Fermat and Euler is the following: given a positive definite quadratic form

$$f[x] = \sum_{i,j=1}^{2k} f_{ij} x_i x_j$$

with integral f_{ij}, how many solutions has the equation

$$(1) \qquad\qquad f[x] = g$$

for an integral g and integral x_i? The question can be put into the frame of analytic function theory by the evident identity

$$(2) \qquad \vartheta(z, f) = \sum_{x \in \mathbf{Z}^{2k}} e^{2\pi i z f[x]} = \sum_g N(f, g) e^{2\pi i g z}$$

with a complex variable z, where $N(f, g)$ denotes the number of solutions of (1). What makes the problem interesting is the fact that this function is a modular form in z satisfying the functional equations

$$(3) \qquad \varphi\left(\frac{az + b}{cz + d}\right)(cz + d)^{-k} = \pm\varphi(z)$$

where $a, \dots, d$ are integers with the properties

$$(4) \qquad\qquad ad - bc = 1, \qquad c \equiv 0 \mod q$$

and where q is an integer, characteristic for $f[x]$, the so-called *level* of f. The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying (4) form the subgroup $\Gamma_0(q)$ of the modular group $\mathrm{SL}_2(\mathbf{Z})$.

Modular forms and their quotients, modular functions, are intrinsically connected with elliptic functions and so belong to the central subjects of