being the excellent books they are, I obtained copies before they were offered to me for review. Ah well!

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Multiphase averaging for classical systems, with applications to adiabatic theorems by Pierre Lochak and Claude Meunier. (Translated by H. S. Dumas), Applied Mathematical Sciences, vol. 72, Springer-Verlag, New York, Berlin, Heidelberg, 1988, xi + 360 pp., \$39.80. ISBN 0-387-96778-8

The idea of a separation of scales is of fundamental importance in our attempts to understand the world. When we speak of movement up or down "on the average," we are appealing to a process which removes rapid fluctuations and uncovers underlying trends. The formal perturbation procedure known as the method of multiple scales (or, in its simplest form, two-timing) relies on such a separation of time scales, as do the various averaging and homogenization theorems which make up an important part of the theory of differential equations and which form the subject of the book under review.

The simplest form of averaging, over a single time scale, proceeds as follows. Starting with a sufficiently smooth vector field $f(x,t)$ on $\mathbf{R}^n \times \mathbf{R}$ which depends T-periodically on time, t, the *averaged vector field* is defined as

$$\overline{f}(x) = \frac{1}{T}\int_0^T f(x,t)\,dt. \tag{0}$$

Averaging theory provides links between the solutions of an "original" ordinary differential equation,

$$\dot{x} = \varepsilon f(x, t), \qquad 0 < \varepsilon << 1, \tag{1}$$

and its averaged counterpart

$$\dot{y} = \varepsilon \overline{f}(y). \tag{2}$$

Here, since ε is a small parameter, the x variables evolve *slowly* in comparison with the *fast* time, t.

The most basic result is that if solutions $x(t)$, $y(t)$ of (1) and (2) are started at $t = 0$ within $\mathscr{O}(\varepsilon)$, they remain within $\mathscr{O}(\varepsilon)$ for a time interval of $\mathscr{O}(1/\varepsilon)$: specifically, one obtains estimates such as

$$|x(t) - y(t)| \leq c\varepsilon(1 + e^{c\varepsilon t}), \tag{3}$$

where c is a constant depending upon f and the initial separation $|x(0) - y(0)|$. Estimates like (3) are easily obtained by introducing a near identity transformation $x = y + \varepsilon u(y, t, \varepsilon)$ which takes (1) to the system

$$\dot{y} = \varepsilon \overline{f}(y) + \varepsilon^2 g(y, t, \varepsilon) \tag{4}$$

and comparing the solutions of (2) and (4) via Gronwall's inequality. Krylov and Bogoliubov [**1**] appear to have been among the first to provide a rigorous treatment of such "single frequency" averaging. Bogoliubov and Mitropolski [**2**] provide a good introduction to and review of the basic results.

Estimates like (3) are useful, for they permit one to draw conclusions regarding the solutions of the nonautonomous equation (1) from those of the autonomous one (2). For example, if (2) has a "nondegenerate" (hyperbolic) fixed point y_0 (i.e., if $\overline{f}(y_0) = 0$ and $D\overline{f}(y_0)$ has no eigenvalues with zero real part, then it follows that (1) has a small ($\mathscr{O}(\varepsilon)$) T-periodic orbit near y_0. Finding a zero of a vector field is relatively easy, while solving for a periodic orbit directly is usually very difficult.

Various generalizations of the basic result sketched above immediately suggest themselves, among them are:

1. Relaxation of time periodicity: f quasi-periodic, almost periodic.
2. Improvement of timescale for estimates such as (3) to $\mathscr{O}(1/\varepsilon^k)$ or even infinite time scales.
3. Improvement of order of estimates.
4. Related methods: adiabatic "invariants"; normal form theory.
5. Extension to infinite dimensional evolution equations.

The book under review explores the first and fourth of these areas at length, and gives a briefer account of some results in the second. It remains in finite dimensions, and so avoids the fifth. As the authors note, the third area was dealt with rather well in another recent book on averaging by Sanders and Verhulst [**3**]). The present book is especially welcome since it includes readable and reasonably complete accounts of recent Russian work which is not as well known in the West as it deserves to be. In particular, multifrequency averaging results of Arnold (cf. [**4**]) and Neistadt are described, as is the important theorem on the variation of action variables in perturbations of integrable Hamiltonian systems due to Nekhoroshev [**5**]. The authors, correctly in my view, point out that this result is a "converse" of the celebrated Kolmogorov-Arnold-Moser (KAM) theorem on "regular" motions and should be much better known. This material occupies the first seven chapters of the book: the last three are devoted to adiabatic theorems. While the final chapter concerns quantum effects, the main emphasis of the book is on classical and Hamiltonian mechanical problems.

Adiabatic theorems again involve a separation of time scales, but here there is typically a slowly varying parameter $\lambda(\varepsilon t)$ in comparison with the relatively rapid state variables $x(t)$ and one considers a differential equation of the form

$$\dot{x} = f(x, \lambda(\varepsilon t)), \tag{5}$$

and seeks a function $A(x, \lambda)$ which is "almost invariant," so that, for example, for most initial conditions $x(0)$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0, 1/\varepsilon]} |A(x(t), \lambda(\varepsilon t)) - A(x(0), \lambda(0))| = 0. \tag{6}$$

A prototypical example is the linear oscillator with slowly varying frequency,

$$\ddot{x} + \omega^2(\varepsilon t) x = 0, \tag{7}$$

in which case the *action* or area enclosed by a level set of the "frozen" Hamiltonian function

$$\frac{\dot{x}^2}{2} + \frac{\omega^2(\varepsilon t)}{2} = \text{const} \tag{8}$$

is the adiabatic invariant.

The flavour of the present book is very different from that of other recent books on averaging and dynamical systems, such as [**3** or **6**]. After a useful general introduction, the mood is established in the second chapter which describes a rather general result due to Anosov on multifrequency (quasiperiodic) systems of the form

$$\begin{aligned} \dot{I} &= \varepsilon f(I, \varphi, \varepsilon), \\ \dot{\varphi} &= \omega(I, \varphi) + \varepsilon g(I, \varphi, \varepsilon), \end{aligned} \tag{9}$$

where I and φ are $m \geq 1$ and $n \geq 1$ dimensional slow and fast variables respectively and f, ω, g and their first derivatives are uniformly bounded. The corresponding averaged system is defined under the assumption that the unperturbed ($\varepsilon = 0$) flow is ergodic (in φ) for almost all I values, and estimates on the closeness of averaged and actual solutions are obtained for sets of initial data whose measure approaches one as ε goes to zero. This measure theoretic viewpoint pervades the book: most of the multifrequency theorems require the exclusion, or special treatment, of a set of "resonant" solutions or actions, on which the averaged and actual solutions diverge more quickly than in other parts of the phase space. In a sense, the remaining eight chapters explore the sharper results available as one puts more specific restrictions on f, ω and g in (9).

It follows that the book is analytical, rather than geometrical, in flavor, although geometry does enter neatly in addressing the extension of time scales and in the problem of resonance capture. Proofs of typical theorems involve technical lemmas in which sets of "bad" solutions must be excluded and their measure estimated, or in which the time spent by solutions in resonant regions must be bounded. Occasional simple examples are given to demonstrate the optimality of results, or to illustrate the theme of a chapter, but the book is most valuable for its collection of technical results not easily accessible elsewhere, and for the relative completeness of its treatment. For the sake of the latter, the authors sometimes give a result in a less general or powerful form than in the literature they quote. This seems especially wise in the case of Nekhoroshev's theorem: the enthusiast can always go to the original literature, and the bibliography is reasonably complete in this respect, although it does not refer to another recent publication [7] which contains many examples of multifrequency averaging methods and which would be a useful (and inexpensive) complement to the present volume.

There are a number of useful appendices on such topics as Fourier series, Diophantine approximations, Hamiltonian systems, Lie series and normal forms, but I was a little surprised that the latter were not more integrated into the body of the text. Averaging theory is, in a sense, a special case of the more general normal form transformation method, in which one successively eliminates "nonresonant" terms at increasing order in some small parameter and then studies the transformed, truncated ($\sim$ averaged) system with a view to obtaining results on the original system (cf. [**8**]). In this respect readers may wish to note that many of the transformation procedures of normal form and averaging theory can now be automated using symbolic manipulation packages such as REDUCE, MACSYMA or SMP (see, e.g. [**9**]).

Another direction which the present book does not take, apart from a brief discussion in Chapter 10, is the "geometrisation" of averaging methods in the context of fiber bundles (roughly speaking, in (9) the slow variables, I, form the base and the fast ones, φ, the fibers). The recent work of Marsden, Montgomery, Ratiu, et. al. [**10**], including their development of the notion of "Berry's phase" [**11**] is especially exciting in this respect.

However, it is ungenerous and irrelevant to complain at things left out when so much is included. This should be a useful book for anyone interested in differential equations and dynamical systems.

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Spectral theory of linear differential operators and comparison algebras by H. O. Cordes. London Mathematical Society Lecture Notes Series, vol. 76, Cambridge University Press, Cambridge, New York, Melbourne, 1987, ix + 342 pp., $29.95. ISBN 0-521-28443-0

Since the introduction of pseudodifferential operators (psdo) in the foundational papers by J. Bokobza and A. Unterberger **[BU]** and by J. J. Kohn and L. Nirenberg **[KN]** more than 20 years ago, the psdo proved to be a powerful tool in the analysis of partial differential operators (pdo) on compact smooth manifolds and euclidean spaces.

Recently much of attention has been shifted to pdo on noncompact manifolds (cf. **[CGT, D3, M, P, R, S]**). It is conspicuous however how little the psdo have been used in this context (cf. **[E]**), possibly because a necessary global symbolic calculus is still in its development. The Cordes book presents a principal calculus of such sort in a C^*-algebras framework.

One of two historical sources of psdo was the theory of boundary value problems for elliptic equations (another was quantization). It was concerned with classical potential representations of their solutions. The potential densities satisfy singular integral equations on the boundary, and a general technique (proposed by G. Giraux in 1934) was a reduction to regular Fredholm integral equations. In 1936 S. G. Mikhlin found a key for such regularization, introducing the (principal) symbol of singular integral operators (sio). Actually he worked with sio on the plane, but his symbol construction was immediately extended by G. Giraux to any euclidean space. The construction was based on a rather heavy decomposition of the sio into multiple power series $\Lambda_j = (-\Delta)^{1/2}\partial/\partial x_j$ of Riesz operators. In the 1950s A. Calderón and A. Zygmund discovered a much more flexible