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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Spectral theory of linear differential operators and comparison algebras by H. O. Cordes. London Mathematical Society Lecture Notes Series, vol. 76, Cambridge University Press, Cambridge, New York, Melbourne, 1987, ix + 342 pp., $29.95. ISBN 0-521-28443-0

Since the introduction of pseudodifferential operators (psdo) in the foundational papers by J. Bokobza and A. Unterberger **[BU]** and by J. J. Kohn and L. Nirenberg **[KN]** more than 20 years ago, the psdo proved to be a powerful tool in the analysis of partial differential operators (pdo) on compact smooth manifolds and euclidean spaces.

Recently much of attention has been shifted to pdo on noncompact manifolds (cf. **[CGT, D3, M, P, R, S]**). It is conspicuous however how little the psdo have been used in this context (cf. **[E]**), possibly because a necessary global symbolic calculus is still in its development. The Cordes book presents a principal calculus of such sort in a C^*-algebras framework.

One of two historical sources of psdo was the theory of boundary value problems for elliptic equations (another was quantization). It was concerned with classical potential representations of their solutions. The potential densities satisfy singular integral equations on the boundary, and a general technique (proposed by G. Giraux in 1934) was a reduction to regular Fredholm integral equations. In 1936 S. G. Mikhlin found a key for such regularization, introducing the (principal) symbol of singular integral operators (sio). Actually he worked with sio on the plane, but his symbol construction was immediately extended by G. Giraux to any euclidean space. The construction was based on a rather heavy decomposition of the sio into multiple power series $\Lambda_j = (-\Delta)^{1/2}\partial/\partial x_j$ of Riesz operators. In the 1950s A. Calderon and A. Zygmund discovered a much more flexible

Fourier transform characterization of sio and their symbols. Of equal importance was the factorization of pdo as $S(-\Delta)^{m/2}$ where S is an sio (cf. **[CZ]**). The power of it was immediately demonstrated by the famous paper by Caldéron **[C1]** on the uniqueness of the Cauchy problem for elliptic equations. A next natural step was to investigate the algebra generated by $S(-\Delta)^{m/2}$ (with arbitrary sio S and m) and derive its principal symbolic calculus. This has been done in **[D1]** with the explicit purpose of providing more homotopy freedom for elliptic boundary problems fortifying singificantly the I. Gelfand strategy to evaluate their indices. Soon J. Bokobza and A. Unterberger **[BU]**, J. J. Kohn and L. Nirenberg **[KN]** and immediately L. Hörmander **[H]** refined the symbolic calculus of (classical) psdo, in particular for study of degenerate boundary value problems. Simultaneously a relationship with H. Weyl quantization showed that the correspondence of psdo to refined symbols is not unique. On the other hand, I. Gohberg **[G]** had already discovered the intrinsic nature of the principal symbol: it is the Gelfand transform of the quotient C^*-algebra Ψ_0 generated by the classical zero order psdo on R^n modulo its commutator. Actually he assumed that the generators are the Riesz operators and multipliers with compactly supported continuous functions. Then the commutator ideal E coincides with the ideal K of all compact operators on $L^2(R^n)$. This is also true for C^*-algebras $\Psi_0(\Omega)$ generated by compactly supported psdo on smooth manifolds Ω. The traditional construction of psdo on manifolds is based on their pseudolocalization to the psdo on coordinate charts. It is rather awkward for global considerations unless the psdo can be approximated with compactly supported operators. In particular the Fredholm properties are difficult to verify. However a global study is possible for the comparison C^*-algebras. This concept of Cordes has been a subject of his work and the work of his collaborators and students for more than 20 years. The book is a fair account of their research. A comparison C^*-algebra A on $L_2(\Omega, d\mu)$ (where $d\mu$ is a smooth positive measure on a noncompact manifold Ω) is generated by smooth multipliers and generalized Riesz operators. The multipliers belong to a class $A^\#$ of smooth bounded functions on Ω. The generalized Riesz operators are of the form $D\Lambda$ where $\Lambda = (-H)^{1/2}$ is the Friedrichs extension of a fixed selfadjoint second order elliptic differential operator ≥ 0 (usually ≥ 1) and D belongs to a class $D^\#$ of vector fields on Ω. A basic example of H is the Laplacian on a Riemannian Ω. In this case A is called a Laplace comparison algebra. An order m pdo P is said to be within the reach of A if $P = S\Lambda^m$ where S is in the comparison algebra. Under general assumptions A contains the ideal K of all compact operators on $L^2(\Omega, d\mu)$, and $H^s = \Lambda^s L^2(\Omega, d\mu)$ form a Sobolev scale, so that Fredholm properties of A on H^s are equivalent to Fredholm properties of S in A. The Gelfand space M of the quotient of A modulo its commutator E contains the Gelfand space M_0 of the ideal Ψ_0/K which is homeomorphic to the cosphere bundle $S^*\Omega$. The space is called the symbol space and its complement $M - M_0$ is the secondary symbol space. The latter is an origin of the essential spectrum of elliptic pde within the reach of the algebra. The book describes a variety of examples of the secondary symbol spaces.

Another theme is the case when $E \neq K$, but $E/K = C(N, K(h))$, the C^*-algebra of continuous functions on N with values in compact operators on a Hilbert space h, and N is the symbol space of a comparison algebra. The $K(h)$-valued functions are the E-symbols; they can be extended to $L(h)$-valued symbols on A. An important example is the Laplace comparison algebra on complete manifolds with finite number of cylindrical ends. In general there are many nonclassical situations (e.g. on noncompact manifolds or manifolds with singularities) when the commutator ideal E of natural C^*-algebras of sio is larger than the ideal K of compact operators, but there exists a finite composition series of ideals $O = J_{-1} < J_0 = K < \cdots < J_r = A$ such that the subquotient J_j/J_{j+1} are continuous trace C^*-algebras with finite-dimensional Hausdorff spectra Σ_j. Such algebras are called solvable and in somewhat less generality have been introduced with many examples in the paper **[D2]** of 1978.

There is a Fredholm and index hierarchy in solvable C^*-algebras which converts inversion of sio modulo ideals J_j into a successive inversion of Riesz-Shauder operator families (cf. **[D2]**). A solvable C^*-algebra of sio (with $r = 2$) was considered by H. Cordes **[C2]** already in 1969. Recently it was shown (cf. **[D3]**) that controlled classical sio generate solvable C^*-algebras on an extensive class of Riemannian manifolds with stratified Thom compactifications.

The Cordes book provides a timely introduction to the rapidly developing field.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Iwasawa theory of elliptic curves with complex multiplication, by Ehud de Shalit. Perspectives in Mathematics, vol. 3, Academic Press, Orlando, 1987, ix + 154 pp., $19.50. ISBN 0-12-210255-x

One of the most fascinating aspects of number theory and arithmetic algebraic geometry is the deep and mysterious connection between arithmetic and analysis. One example of this is the formula for the residue of the zeta function of a number field F,

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{d}} \tag{1}$$

where r_1 (resp. r_2) is the number of real (resp. complex) embeddings of F, h is the class number of F, R is the regulator (a determinant of logarithms of global units) of F, w is the number of roots of unity in F, and d is the discriminant of F.

Another, more modern example deals with elliptic curves. If E is an elliptic curve defined over a number field F (i.e., E is a curve defined by an equation $y^2 = x^3 - ax - b$ with $a, b \in F$ and $4a^3 - 27b^2 \neq 0$), then E has an L-function and various arithmetic invariants. The fundamental object of arithmetic interest is the set $E(F)$ of points on E with coordinates in F; $E(F)$ has a natural abelian group structure and by the Mordell-Weil theorem this group is finitely generated. The L-function is defined by an Euler product over primes $\mathfrak{p}$ of F,

$$L(E,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\mathbf{N}\mathfrak{p}^{-s})$$

where $L_{\mathfrak{p}}(T)$ is a polynomial in T of degree at most 2, whose coefficients depend on the reduction of E modulo $\mathfrak{p}$. The conjecture of Birch and Swinnerton-Dyer states that

$$\mathrm{rank}_{\mathbf{Z}} E(F) = \mathrm{ord}_{s=1} L(E,s) \tag{2}$$

and further, if we denote this common value by r, the conjecture expresses $\lim_{s\to 1}(s-1)^{-r}L(E,s)$ in terms of other invariants of E, with a formula analogous to (1).