none of these really detracts from the mathematical merits of the book. In particular, all is forgiven when you discover in §1 of Chapter 8 what is perhaps the most intriguing remark in the whole book, which reads in bold face "**1.7. No Remark**"!

Summing up, Scharlau has presented to the mathematical community a new book in quadratic forms which is well written, eminently readable and of excellent reference value. While obviously many important topics have to be left out, the selection and organization of the material overall were done thoughtfully and with good vision. The end-product is a book which succeeded for the first time in encompassing the theories of quadratic and Hermitian forms, with a very substantial and engaging coverage of both. A book like this usually has a very positive effect on the growth of a field. Obviously, researchers and students alike in the area of quadratic forms should thank Scharlau for his great effort.

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Collected mathematical papers of Jacob Nielson, edited by Vagn Lundsgaard Hansen. Birkhäuser, Boston, Basel, Stuttgart, 1986, vol. 1, 1913–1932, xix + 458 pp., sFr. 198. ISBN 0-8176-3140-2, vol. 2, 1932–1955, xi + 440 pp., sFr. 192. ISBN 0-8176-3151-8

Introduction. This valuable two volume set *Jakob Nielsen: Collected mathematical papers* contains thirty seven articles originally published between 1913 and 1955. Several of the papers appear for the first time in translation from the original German and Danish. The volumes also contain a brief biographical sketch and five essays on the impact of Nielsen's work.

The five essays review Nielsen's work in four different fields. The essay by Werner Fenchel outlines Nielsen's contributions to the theory of discontinuous groups of isometries of the hyperbolic plane and the geometry of the surfaces determined by such groups. Bruce Chandler and Wilhelm Magnus outline the purely group theoretic results of Nielsen and describe their influence on the subsequent development of combinatorial group theory. Robert F. Brown traces the development of Nielsen fixed point theory, and essays by Joan Birman and Heiner Zieschang discuss Nielsen's work on surface mapping-classes.

The Danish Mathematical Society was interested in publishing the collected works for a long time. W. Thurston's preprint, *On the dynamics of diffeomorphisms of surfaces*, I began to circulate in 1976. Thurston's work sparked a renewed interest in Nielsen's work. Presumably this renewed interest gave important impetus to the Danish Mathematical Society's project and for the project to include translating some of the papers, especially [1-I, II, & III].

The relation between Nielsen's work and Thurston's work raises interesting questions about the nature of mathematical progress and the nature of mathematical discovery. Thurston's preprint was not published until this last October, more than ten years after it renewed interest in Nielsen's work. The fact that the *Bulletin of the AMS* has just published Thurston's paper makes this an appropriate time and setting to consider the impact of Thurston's work upon how mathematicians working in Riemann Surfaces and Teichmüller theory regarded Nielsen's work, especially [**1** and **2**].

A question. The question of how much of the Thurston theory Nielsen knew is frequently asked. I quote from a letter I received last May from a recent Ph.D.:

> "I would like to ask you some questions about Nielsen's contribution to the problem:
>
> (1) Did Nielsen know these results or at least part of them? You wrote... that if a lift of a pseudo-Anosov diffeomorphism to the unit disk, U, fixed n-pronged singularity then it has $2n$ fixed points on the boundary of U. Unfortunately... I also could not find this result among Nielsen's papers.
>
> (2) The most difficult part of the proof of Thurston's theorem is the case of two fixed points on the boundary of the unit disk. Did Nielsen know that then a lift of a pseudo-Anosov diffeomorphism does not have fixed points inside U?
>
> (3) And finally what did Nielsen know about the dynamic behavior of lifts of pseudo-Anosov diffeomorphisms?"

Although he would not have phrased it that way, Nielsen was studying the dynamic behavior of diffeomorphisms of surfaces, including the dynamic behavior of the lifts to the unit disk of pseudo-Anosov diffeomorphisms. While the word pseudo-Anosov was not in his vocabulary, he did know a great many of the pieces of the Thurston theory. He produced essentially all of the technical results needed; however, he lacked the conceptual framework that gives Thurston's work its impact.

To illustrate this more concretely I will discuss Nielsen's work on completely reducible mapping-classes and his investigation of pseudo-Anosov mapping-classes.

Completely reducible mapping classes and the connection. As many mathematicians working in the field of Riemann Surfaces and Teichmüller theory and in particular working on mapping-class groups did, I spent some time reading various papers of Nielsen, in particular the papers on finite order homeomorphisms and surface mapping class groups of algebraically finite type [**2**].

In the Spring of 1976 I heard Thurston deliver a series lectures on his classification of surface diffeomorphisms. Two years later during the 1978 Stony Brook conference Bill Harvey suggested a problem to me that involved trying to use techniques I had developed to find conjugacy invariants for one of Thurston's type of diffeomorphisms, namely those that induced what are now called completely reducible mapping-classes (reducible diffeomorphisms all of whose component maps are homotopic to maps of finite order). It was while I was working on this problem that I realized that finding conjugacy invariants for such mapping-classes was precisely what Nielsen had been doing in [**2**] and at that point I obtained copies of [1-I, II, and III] and began to read them. What I discovered was that Thurston's classification could be derived using Nielsen theory.

Many people at that time had read Nielsen. When I reported to Bill Harvey that Nielsen had been finding the conjugacy invariants that he had suggested I look for, his reaction was "Oh, of course." The point is that although Nielsen's work was widely known and studied, nobody made much sense or purpose of it. Everyone had a sense that it was important, but it seemed unfocused and unmotivated. However, Thurston's point of view made it instantly clear to us what Nielsen was doing.

A summary of the Thurston and Nielsen theories. For ease of exposition I restrict the discussion to compact Riemann surfaces, S, of genus $g \geq 2$. The mapping class group of S, M_g, is the group of homotopy (or equivalently in this case, isotopy) classes of diffeomorphisms of the surface.

Thurston's classification of surface diffeomorphisms states that a diffeomorphism which is not homotopic to a periodic diffeomorphism is homotopic to either one which fixes a partition on S or to one which fixes a pair of transverse measured foliations.

A diffeomorphism that fixes a partition on S is called **reducible** and one that fixes a transverse pair of measured foliations is called **pseudo-Anosov**.

A **partition** on S is a set of disjoint simple closed curves. Roughly speaking a **measured foliation** is a family of pairwise disjoint geodesic lines (known as *leaves*) on S. A pair of transverse measured foliations is **fixed** by the diffeomorphism if the leaves are fixed by the transformation so that along one set of leaves the diffeomorphism expands by a fixed factor and along the other it contracts by that factor.

A mapping-class or homotopy class is pseudo-Anosov or reducible if it contains a representative that is pseudo-Anosov or reducible.

To describe the classification using Nielsen theory, let h be a diffeomorphism of S, H the group generated by h, and $L(H)$ the groups of lifts of elements of H, to U, the unit disk. Each lift of a diffeomorphism to U extends to the boundary of U. Nielsen assigns a pair of integers to each lift. Roughly speaking the integers signify the number of certain orbits of attracting and repelling fixed points of the homeomorphism on the boundary and the number of certain orbits of the automorphism induced on the fundamental group of S by the diffeomorphism. The pair of integers, (u_g, v_g), which is known as the **Nielsen type** of the lift, $g \in L(H)$, thus describe the dynamical behavior of the lift. Thurston's classes can be defined by describing the Nielsen types occurring in $L(H)$. For example, a pseudo-Anosov mapping-class corresponds roughly speaking to a diffeomorphism h for which most $g \in L(H)$ have $v_g = 0$.

Nielsen's view of pseudo-Anosov maps. Nielsen studied some pseudo-Anosov diffeomorphisms. In fact he spent a great deal of time studying one particular pseudo-Anosov diffeomorphism, known as Example #13. Nielsen did not define the class of pseudo-Anosov diffeomorphisms; however, he did have a sense that there was something important about example #13. Example #13 maps a surface of genus 2 into itself and if the fundamental group of the surface has presentation

$$\langle a, b, c, d | [a, b][c, d] = 1 \rangle,$$

then diffeomorphism #13 is induced by the automorphism

$$a \to c^{-1}a^{-1}, \qquad b \to b^{-1}a^{-1}, \qquad c \to b^{-1}a^{-1}d, \quad \text{and} \quad d \to c^{-1}.$$

Nielsen studied example #13 so extensively in [1-I] and [1-III] that he proved every single property that one needs in order to verify using Thurston theory that this is a pseudo-Anosov mapping-class. Thurston's characterization of a pseudo-Anosov diffeomorphism was the fundamental characterization towards which Nielsen was striving and he certainly would have recognized the significance of the definition of a pseudo-Anosov and that #13 is pseudo-Anosov. As noted above, one can characterize a pseudo-Anosov by the Nielsen types of its lifts. This yields a definition which is equivalent to Thurston's definition but which does not convey the essence of the dynamics of the pseudo-Anosov map. The concept of a foliation is the missing piece.

Thurston's original paper spurred many people to study pseudo-Anosov diffeomorphisms. There are several different methods known for constructing pseudo-Anosov diffeomorphisms. They can often be recognized when written as a product of Dehn twists. However, I have never seen anyone complete the messy calculation necessary to write #13 as a product of Dehn twists and I do not know if any of the currently known methods of construction would produce this example. One would really like to know what prompted Nielsen to write down example #13. If he had explained this to us, we might be able to construct a new class of pseudo-Anosov maps.

Conclusion. One answer to the question of how Nielsen's work is related to Thurston theory is that there is probably no statement in Thurston's work that would have surprised Nielsen. Nielsen's work contains almost all of the technical results needed to produce Thurston's classification, but Nielsen lacked the conceptual framework needed to complete the task. However, Nielsen's techniques allow one to answer quickly many questions that Thurston's classification naturally leads one to ask. Numerous papers written in the last ten years have used Nielsen's techniques to answer such questions.

The complexity and cumbersome nature of Nielsen's work highlight the clarity and elegance of Thurston's intuition. Conversely, Thurston's work makes us appreciate the mathematical insight behind the technically difficult work of Nielsen. Because of Thurston's work, Nielsen's work continues to prove new and important theorems and to have a role in current mathematical life and thought.

The focus of this essay should not detract from Nielsen's varied and fundamental accomplishments. Because of the depth and breadth of Nielsen's work, all of the five essays in the *Collected mathematical papers* are needed to do full justice to it. Nielsen made fundamental contributions in several areas which are now considered distinct fields. Many terms, concepts, theorems, and even a whole field bear his name. The names *Nielsen transformation, the Nielsen number, the Nielsen realization problem, the Nielsen-Schrier theorem, the Dehn-Nielsen theorem, Nielsen fixed point theory* begin to illustrate his influence. Further Nielsen made equally fundamental discoveries that do not bear his name. For example, Nielsen published in Danish a theorem that is important in dynamical systems and which was found independently by Marston Morse: on every closed surface with constant negative curvature there exist geodesics which are everywhere dense on the surface and approach every direction. Many of Nielsen's technical advances which were very deep results for the time have become basic results to modern mathematicians. The Danish Mathematical Society deserves our sincerest gratitude for translating and publishing these important papers.

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