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Basic hypergeometric series and applications, by Nathan J. Fine. Mathematical Surveys and Monographs, vol. 27, American Mathematical Society, Providence, R.I., 1988, xiii + 124 pp., $39.00. ISBN 0-8218-1524-5

Part of the beauty of mathematics is the strange way an elementary problem or result can be slightly extended and all of a sudden a whole new set of much deeper problems and results arise. The subject of explicit evaluation of integrals is one example. In advanced calculus we introduce the gamma function either as an infinite product or through Euler's integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{1}$$

and use it to evaluate the beta integral

$$\int_0^1 t^{x-1}(1-t)^{x-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2}$$

or the equivalent integral

$$\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

In the seventeenth century when elementary integration was being developed, the first evaluation of the integral of x^k for all rational k used a geometric dissection of an interval, that is the measure $aq^n(1-q)$ was put at the point $x = aq^n$, $n = 0, 1, \ldots$. This is an approximation to the uniform distribution on $[0, a]$ to which it reduces when $q \to 1$. Eventually Ramanujan found an extension of (3) using this measure for fixed q. This integral is really an infinite series which is usually written as

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n \tag{4}$$

where $|q| < 1$, $|b/a| < |x| < 1$, and

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{5}$$

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty. \tag{6}$$

The present book treats in detail the function

$$F(a,b;t;q) = \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n, \qquad |t| < 1. \tag{7}$$

This is a section of the sum (4), and so can be thought of as an extension of the incomplete beta integral

$$\int_0^t \frac{x^a}{(1+x)^b}\,dx. \tag{8}$$

This integral seems very special and outside of some statisticians and a few applied mathematicians and physicists, almost no one else would think that it occurs in the solution of interesting problems. The sum (7) thus seems to be an extension of something that itself is not very interesting, and so of marginal interest. Surprisingly that is not the case.

To illustrate the surprising topics that arise when treating (7) in detail, here are two, the rank of a partition and mock theta functions. The rank of a partition was discovered by Freeman Dyson when he was an undergraduate at Trinity College, Cambridge. A partition π of an integer n is a set of positive integers $(\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\lambda_1 + \cdots + \lambda_r = n$. This is often given as a set of dots

$$\begin{array}{ll} \lambda_1 & \bullet\bullet\bullet\bullet\bullet \\ \lambda_2 & \bullet\bullet\bullet \\ \lambda_3 & \bullet\bullet \\ \lambda_4 & \bullet\bullet \\ & \bullet \\ & \bullet \\ & \bullet \\ \lambda_r & \bullet \end{array}$$

with λ_1 in the first row, λ_2 in the second and so on. The rank of a partition is $\lambda_1 - r$, the number of dots in the first row minus the number of rows. If $p(n)$ denotes the number of partitions of n, Ramanujan observed and then proved that $p(5n+4) \equiv 0 \pmod 5$. Dyson looked for a way of decomposing the partitions of $5n+4$ into 5 equinumerous classes, and discovered empirically that the rank of a partition provides such a splitting. This was later proved by Atkin and Swinnerton-Dyer. There are interesting generating functions for the rank, and other combinatorial facts that are obtained in this book. For example, consider the rank of partitions mod 4. There are always more partitions of $2n$ whose rank is congruent to 1 mod 4 than there are partitions of $2n$ whose rank is congruent to 2 mod 4, and the difference is the number of partitions of $2n$ into odd parts without gaps, i.e. if $2j+1$ and $2k+1$ occur, then $2l+1$ occurs for all l between j and k, and parts may be repeated. A similar result is true for partitions of odd integers, so for mod 4 there is never a result like Ramanujan's for $p(5n+4)$ mod 5 that can be explained by the rank of a partition. Since the rank works for Ramanujan's result $p(7n+5) \equiv 0 \pmod 7$, but does not work for $p(11n+6) \equiv 0 \pmod{11}$, one is not completely surprised that there is a partition interpretation for the difference of the above two classes of partitions, but it is far from clear a priori that such will exist.

Mock theta functions have been a mystery ever since Ramanujan wrote about them in his only letter sent to Hardy after Ramanujan returned to India. Mock theta functions are functions in the unit circle whose boundary behavior can be computed to an accuracy almost as good as one can do for theta functions using a modular transformation, yet they are not reducible to a finite sum of theta functions. Ramanujan gave a more precise definition, and listed examples of third, fifth and seventh orders. The four of third order that he listed can all be transformed into series that are special cases of $F(0, t^{-1}; t; q)$. For example, Ramanujan's examples $f(q)$, $\phi(q)$ and $\psi(q)$ were defined originally by

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \\ \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} \\ \psi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \end{aligned}$$

Fine shows that

$$\begin{aligned} f(q) &= 2F(0,-1;-1;q) \\ \phi(q) &= (1-i)F(0,-i;i;q) \\ \psi(q) &= -1 + F(0,q^{-1};q;q^4). \end{aligned} \tag{9}$$

Formula (9) and a functional equation given earlier allows Fine to obtain some very surprising congruences. For example, if $L_k(n)$ denotes the number of partitions of n into distinct parts with smallest part equal to k, then

$$(-q;q)_\infty f(q) = 1 + 2\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} L_{2k+1}(n)\right) q^n = 1 + 2\sum_{n=1}^{\infty} L(n)q^n.$$

If p is an odd prime, $\alpha > 0$, then $L(p^{2\alpha}) \equiv 1 \pmod 4$, $L(p^{2\alpha-1}) \equiv 0 \pmod 4$ if $p \geq 7$ and $L(3^{2\alpha-1}) \equiv L(5^{2\alpha-1}) \equiv 2 \pmod 4$.

There are many other interesting results in this book. The history of the book is also interesting. It is a slightly revised version of a manuscript Fine wrote years ago. Many of his results were obtained in the late 1940s and 1950s. He lectured on some of this as the Hedrick Memorial Lecturer to the MAA in 1966. The largest change in the present book from an earlier manuscript (which was not widely circulated) is the notation, which is now the standard notation of basic hypergeometric series. There are a few new results Fine added, and chapter notes were added by George Andrews. While a lot has been done on this subject in the last twenty years (mostly due to the work of George Andrews), very few of Fine's remarkable results have been rediscovered. Some mathematics appears before its time. In this area, the work of L. J. Rogers is the best example. Fine's book would have been another example if it had appeared twenty years ago. Now I think is the right time for a number of people to appreciate his work. We have Andrews to thank for the appearance of this book, as Fine remarks in his introduction.

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