new to the field will derive full benefit from it after they have had a course that takes them in a leisurely fashion through the more traditional parts of the subject. Alternatively, they may supplement Chapter I by studying the basic material given in Chapter 2 of the author's earlier book with U. Grenander [2].

I cannot conclude this review without referring to a novel and pleasing feature of the book—the fairly detailed and interesting discussion of turbulence, a topic in which the author has been interested for many years. As far as I am aware, this is the first book (in English) on stationary processes to include a treatment of this problem.

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Free rings and their relations, by P. M. Cohn, second edition, Academic Press, London Mathematical Society Monograph No. 19, 1985, xxii + 588 pp., $96.00. ISBN 0-12-179152-1

The fundamental theorem of combinatorial group theory is that a subgroup of a free group is free. In contrast, subalgebras of free (associative) algebras are not well understood, and at the moment defy classification. This is not too surprising: in going from group theory to ring theory the translation of "subgroup" is (one-sided) "ideal." Thus the correct, and fundamental, theorem is that one-sided ideals of free algebras are free submodules. This is a consequence of a "weak algorithm" that holds in the free algebra $k\langle X\rangle$ on the basis X over the field k : if $a_1,\dots,a_n$, $b_1,\dots,b_n$ are homogeneous elements of $k\langle X\rangle$ and $\sum a_i b_i = 0$ then some b_i is in the left module generated by the others. The possibility of a weak algorithm can be investigated in any graded ring, and hence in any filtered ring, and the notion can be refined to that of n-term weak algorithm (bound the number of summand in the definition). It is the discovery and investigation of the consequences of weak algorithms which marks the beginning of the subject treated in this monograph. That the algorithms appear in Chapter 2 reflects the fact that the above mentioned consequence is of more fundamental importance. Thus let an n-fir be a ring in which n generator submodules of free modules are free (of unique rank), a semifir be an n-fir for all n, and a fir be a ring in which all submodules of free modules are free. Then n-term weak algorithm $\Rightarrow$ n-fir while weak algorithm $\Rightarrow$fir. For commutative rings, weak algorithm=euclidean, fir=PID and semifir=2-fir=Bezout domain. The basic properties of n-firs etc. are given in Chapter 1, after an excellent preliminary chapter on general notions.

Throughout the text, the precise conditions for a theorem to hold are given, which makes browsing through the book somewhat difficult. For the purpose of this review we will do with easily stated versions.

Chapters 3 and 4 deal with unique factorization theorems. A factorization of the element c of the ring R is chain of cyclic submodules in the lattice of modules between cR and R. A unique factorization theorem states that any two such chains have isomorphic refinements. As usual there are two parts to such a theorem: enough irreducibles (called atoms) and uniqueness. Firs are automatically atomic, but semifirs need not be. However an atomic semifir is a UFD. More is true: define an $n \times n$ matrix A to be full if $A = BC$ implies that B has at least n columns. Remarkably enough, much of the factorization theory for elements goes over to full matrices. The set of full matrices plays a key role in Chapter 7. For free algebras, the factorization theory is even nicer. The lattice of cyclic modules above an element c is distributive. This implies that similar right factors of c actually only differ by a unit. These chapters need particularly careful reading for the subtleties to be appreciated.

Chapter 5 deals with the linear algebra of finitely presented bound modules M over a semifir R (M is bound if $\mathrm{Hom}(M, R) = 0$). If R is a fir, such a module of nonzero Euler characteristic has both ACC and DCC on bound submodules, and thus a Krull-Schmidt theorem holds for finitely presented modules. The torsion modules, i.e. those of characteristic 0 defined by full matrices were dealt with earlier. They actually form an abelian category, a fact which underlies the factorization theorems for matrices. The first embedding of a fir in a (skew) field appears, into the endomorphism ring of a direct limit of modules of characteristic one, and a bit later we see the first embedding of $k\langle X\rangle$ into a field in which all full matrices become invertible. This last proof uses the important "specialization lemma" (whose proof here is correct, while the one in **[1]** is not) and a familiar ultrproduct construction.

Chapter 6 is devoted to free algebras. The high points are Bergman's centralizer theorem, Dicks' pictorial version of Czernakievicz's proof that automorphisms of $k\langle x_1, x_2\rangle$ are tame and his extremely compact proof of the Kharchenko Galois correspondence theorem. This last is worth stating here because it illustrates the fact that given the choice of $k[X]$ or $k\langle X\rangle$, $k\langle X\rangle$ is often a better deal: Let X be finite, let V be the subspace of $k\langle X\rangle$ spanned by V and let G be a finite group of linear transformations of V, so that G acts on $k\langle X\rangle$. Then $\text{fix}(G)$ is a free algebra and there is a natural bijection between the subgroups of G and the free subalgebras of $k\langle X\rangle$ which contain $\text{fix}(G)$.

The more traditional ring theorist who thinks a ring must either have chain conditions or be PI to be interesting may (very wrongly) take the above material to be rather esoteric. The last major chapter should convince him that this is not so.

The problem addressed here is that of classifying all epimorphisms $e: R \to D_e$ of a ring R into a field D_e. This is an ambitious endeavor, since finding even one such nontrivial epimorphism is a difficult problem. Mal'cev in the thirties gave an infinite set of conditions for a ring to be embeddable in a field and gave examples of domains that do not satisfy these conditions. If R is an Ore domain then there is a unique embedding of R into such a D, its field of fractions, but even here it is not clear what the category of all such epimorphisms looks like. There the matter rested till Cohn's marvelous insight: such an epimorphism e is characterized by the set $\mathscr{P}_e$ of matrices over R whose images are singular over D_e, and there is a specialization $D_e \to D_{e'}$ iff $\mathscr{P} \leq \mathscr{P}_{e'}$. The properties of $\mathscr{P}_e$ can be abstracted and a set $\mathscr{P}$ of matrices with these properties, called a prime matrix ideal, can be defined without prior reference to an epimorphism into a field. The main idea is to define an epimorphism into a field D, given such a $\mathscr{P}$. Perhaps the most transparent way of doing this is to think of the elements of the field D as equivalence classes of formal symbols $rA^{-1}c$ where A is a matrix not in $\mathscr{P}$ and r, c are row and column vectors over R. The arithmetic of such symbols is patterned on such formulas as

$$r_1A_1^{-1}c_1 \cdot r_2A_2^{-1}c_2 = (r_1, 0)\begin{pmatrix} A_1 & -c_1v_2 \\ 0 & A_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

The actual proofs that all this works are of necessity quite technical and are likely to put one off at first sight. The first time reader might be advised to concurrently read a less detailed account of the theory, say Chapter 4 of [**1**] or Chapter 1.4 of [**2**].

Now, a nonfull matrix over a field is singular. It follows that if the set of all nonfull matrices is a prime matrix ideal then the corresponding epimorphism $e: R \to U(R)$ is universal: there is a specialization of $U(R)$ to any R-field. The main application of these ideas is to semifirs (actually to Sylvester domains, a larger class). The set of nonfull matrices in a semifir is a prime matrix ideal and thus every semifir R has a universal field $U(R)$.

The equivalence classes which define $U(R)$ are rather difficult, but not impossible, to handle. In particular, the relationship between matrices that appear in the definition of a given element of $U(R)$ is described. Finally, the result of universally inverting a set of matrices (not necessarily a prime matrix ideal) from a semifir is considered. There are some nice results here. For example it is shown that the group algebra of a free group is a fir (there are other proofs) and that the algebra of rational power series is also a fir.

The notes at the end of each chapter give a good account of the history of the subject. The exercises are plentiful, and range from fairly difficult to open problems (the reader is warned of which category he is dealing with).

This text is an invaluable tool for the researcher and the diligent reader will find it quite rewarding. The reader interested in more examples and applications (some spectacular) is directed to Cohn's companion volume [**1**], and especially to Schofield's lovely monograph [**2**]. Both of these give accounts of Bergman's indispensable coproduct theorems.

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Sphere packings, lattices and groups by J. H. Conway and N. J. A. Sloane. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988, xxviii + 663 pp., $87.00. ISBN 0-387-96617-X, and ISBN 3-540-96617-X

Most of us, mathematicians or not, playing with pennies or a compass at an early age, learnt that six circles fit exactly round an equal one; and most of us, whether we have the mathematical language or not, know that you can't do better: each outer circle subtends just one sixth of a revolution at the centre of the inner one. The **kissing number**, in two dimensions, is six.

In three dimensions the situation is much less clear. A theorem of Archimedes tells us that the solid angle subtended by one sphere at the centre of an equal touching sphere is $(2-\sqrt{3})\pi$. Divide this into 4π and get $8+4\sqrt{3}$, so the kissing number in three dimensions is less than 15. But it's clear, when you try to arrange billiard balls round another one, that you have to leave **holes**: you can always stare through the interstices at the central ball. It's not difficult, by taking this into account, to see that the kissing number is less than 14, but to prove that it is less than 13 is far from trivial. Indeed, as eminent mathematicians as David Gregory and Isaac Newton had an inconclusive discussion about it in 1694. The