from what appears to be direct translation from French. The authors are blameless in this, in my view, but Springer should be ashamed not to have done a decent job of copyediting.) In the face of the importance of the subject matter, it has been surprising how difficult it has continued to be to make one's initial foray into Riemannian geometry. While there may be no royal road to geometry, this book offers at least some clear signposts to readers. But its terseness leaves them to walk a great many lonesome valleys by themselves.

REFERENCES

1 J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian geometry*, North-Holland, American Elsevier, New York, 1975.

2 D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math., vol. 55, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

3 N. Hicks, *Notes on differential geometry*, Van Nostrand-Reinhold, London, 1971.

4 W. Klingenberg, *Riemannian geometry*, de Gruyter Studies in Math., no. 1, Berlin-New York, 1982.

5 S. Kobayashi and K. Nomizu, *Foundations of differential geometry* (two volumes), John Wiley, New York, 1969.

6 M. Spivak, *A comprehensive introduction to differential geometry* (five volumes), Publish or Perish, Inc., Boston, 1975.

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The complex analytic theory of Teichmüller spaces, by Subhashis Nag. John Wiley, New York, Chichester, Brisbane, Toronto, Singapore (Canadian Mathematical Society Series of Monographs and Advanced Texts), 1988, xii + 427 pp., $54.95. ISBN 0-471-62773-9

Teichmüller theory often reminds me of a mathematical foundling. It first appeared, albeit in a different guise, in the attempt by Friedrich Schottky to prove that Riemann surfaces (qua algebraic curves) can be uniformized (parametrized) by meromorphic functions defined in plane domains. According to Felix Klein, Weierstrass, as Schottky's thesis advisor, rejected this (quite correct) argument and removed it from the thesis. So Teichmüller theory was almost stillborn. It explicitly appears first in the work of Klein's collaborator, Robert Fricke. The theory was discovered anew by Oswald Teichmüller around 1940 and reached maturity around 1960 under the tender care of Lars Ahlfors, Lipman Bers and their students following pioneering work of Ernie Rauch.

Almost immediately the theory found a series of foster parents. First, algebraic geometers took us, the noble but isolated practitioners of this iconoclastic discipline, under their mightly wings. We learned the joys of providing lemmas solving partial differential and integral equations and various other nuts and bolts results. These served to render provable such theorems as "The ?%#$! functor is representable." After Mumford, Knudsen, Gieseker et al. proved that the moduli space of stable curves is a projective algebraic variety, we were given two weeks notice and severance pay.

Fortunately, our role as homeless waifs was to be short-lived. Having toiled in the field of mostly hyperbolic surfaces, we had also extended our labors to hyperbolic 3-manifolds or, in our language, to Kleinian groups.[1] A new foster home was waiting in the wings. For Bill Thurston, having finished foliating higher dimensional spaces, had started studying hyperbolic surfaces and 3-manifolds. After him, followed the children of topology. Here again a foreign language was used to express results, some of which we were told were our own.

Then came the merger of complex analysis and dynamical systems, modelled on the theory of Kleinian groups and lead by Douady, Hubbard, Sullivan and Thurston. And behold, they needed a deformation theory. So Teichmüller theory, like Lazarus, again rose to assist Sullivan in showing that iteration of rational functions leads to no wandering domains. It provided, as a byproduct proved at various times and levels by Manẽ, Sad and Sullivan **[16]**, Sullivan and Thurston **[19]** and Bers and Royden **[8]**, the main theorem on extending holomorphic families of motions. This most astounding of theorems is now called the *λ-lemma.*

The past thirty years have been kind to Teichmüller theory. Applications have arisen in a wide variety of mathematical areas and it seemed to have developed a stable clientele. Then, in the mid 1980s, some of us started receiving strange calls from physicists. They seem to need Teichmüller theory too.[2] The Teichmüller space seems to play a central role in string theory. It was also the model for the moduli theory of Yang–Mills fields. So we might conclude, as I. M. Gelfand remarked to Lipman Bers, that *Teichmüller theory has tenure in physics.*

Soon it may be appropriate for me to say what Teichmüller theory is all about. First let me place it in the mathematical firmament. Bers once called it *the higher theory of Riemann surfaces.* I never quite understood that comment. I can only speak—forgive me, I mean write—as a practitioner. To work in Teichmüller theory is to work in an active classical discipline which lies at an intellectual crossroads for contemporary mathematics and high-energy physics. We don't seem to be at a loss for problems important either for internal structure or application to external areas.

1. What is Teichmüller theory and where did it come from? In the late 1920s, Helmut Grötzsch **[13]** posed and solved an extremal problem for $\mathscr{C}^\infty$-mappings between two rectangles. He asked for the diffeomorphism f from one rectangle R_1 to the other R_2 which was the most nearly conformal map and, further, which mapped an ordered set $\mathscr{V}_1$ of vertices of R_1 to an

[1] It was really Poincaré who named them.

[2] The physicists seem to think that the study of Riemann surfaces forms a branch of algebraic geometry. To me it is indicative only of the fields of specialization of the people to whom they first spoke.

ordered set $\mathscr{V}_2$ of vertices of R_2. As measure of closeness to conformality he used the highly nonstandard sup norm K_f of the distortion (dilatation) $K_f(z)$ of f. Using complex derivatives,

$$K_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}.$$

Modulo technical hypotheses, orientation-preserving mappings are called *quasi-conformal* if $K(f)$, the $\mathscr{L}^\infty$ norm of $K_f(z)$, is bounded.

Grötzsch's solution presaged the whole theory of extremal length of curve families. The key step in the proof is to compare the (square of the) average length of the image of horizontal lines to the area of the image rectangle. This generalizes to various length-area comparisons that have pervaded conformal geometry ever since. In the mid 1930s, both Ahlfors and Lavrentiev used quasiconformal mappings in some of their analytic studies (in value distribution theory and partial differential equations, respectively).

Around that time, Teichmüller was reading Ahlfors' geometric studies of value distribution theory and wrote a few papers in the field. After Teichmüller moved from Göttingen to Berlin to work with Bieberbach, he became aware of the pioneering work of Max Schiffer on the seemingly intractable Bieberbach conjecture.[3] Schiffer had shown that a class of variations, of the conformal structure of domains, leads to quadratic differentials. In Schiffer's case this meant second-degree, first-order differential equations. Further, the trajectory structure of the differential equation played a major role in determining the nature of the boundaries of domains which appear as the images under mappings which are extremal.

So Teichmüller merged these two ideas. In a series of papers,[4] he first lay down the foundations of the theory of extremal quasiconformal mappings. He then proved that between any two compact Riemann surfaces of the same genus there is a unique mapping f_0 in each homotopy class which minimizes $K(f)$. In the literature, the existence and uniqueness statements are referred to as separate theorems—Teichmüller published them separately. The proof, especially the existence part, was considered quite controversial since virtually no one could read parts of it.

Briefly stated in the case of most interest, Teichmüller started with a fixed compact Riemann surface S_0 of genus g which we assume greater than one to avoid elementary cases. A surface S together with a homeomorphism f from S_0 to S is called a *marked surface* (S, f). Two marked surfaces (S, f) and (S', f') are called *equivalent* if $f' \circ f^{-1}$ is homotopic to a conformal map from S to S'. The *Teichmüller space* (of genus g) is the space of equivalence classes of marked surfaces of genus g and is denoted

[3]In that circle of politicized mathematicians, reference to the work of Jewish mathematicians was virtually never made. However, Schiffer's work, being on Bieberbach's conjecture, could be referenced.

Bieberbach's conjecture is now, of course, the magic theorem of De Branges (see [**9** or **11**]).

[4]Teichmüller's papers are most easily found in his collected works [**20**].

T_g. It carries the (topological) Teichmüller metric

$$d_T(S, S') = \inf \log K_f$$

where the infimum is taken over all homeomorphisms of S to S' in the homotopy class of $f' \circ f^{-1}$.

In 1954, Ahfors [4] cleaned up Teichmüller's uniqueness proof and gave a new, technically awesome, proof that the extremal exists. Over the next few years, it became apparent that a central role in the deformation theory of Riemann surfaces is played by the solutions to the Beltrami equation

$$f_{\bar{z}} = \mu(z) f_z \tag{1}$$

in **C** with μ measurable and $dj\mu dj_\infty < 1$. μ is then called a *Beltrami coefficient*. The theorem which gives the solution might aptly be called the measurable Riemann mapping theorem; it was proved by Morrey [17] in 1938. The theorem which is usually called the measurable Riemann mapping theorem is due to Ahlfors and Bers [5] in 1960. Their approach to the problem demonstrates that a normalized solution f to Equation 1 depends analytically on those parameters on which μ depends on analytically. In most recent applications one needs more than just the points provided by Morrey's theorem. The analytic structure of the space of solutions is provided by the Ahlfors-Bers theorem.

Given any quasiconformal mapping f from S to S', we may compute partial derivatives with respect to a local coordinate z. Then $\mu_f := f_{\bar{z}}/f_z$ is a $(-1, 1)$-form on S and μ_f may be lifted to a Beltrami coefficient μ on the upper half plane U. μ can be extended to **C** by reflection (set it equal to zero on **R**). The symmetry in μ leads to a symmetry in the solution to the Beltrami equation $w_{\bar{z}} = \mu w_z$ in **C**. The group G of deck transformations for the covering $\pi : U \to S$ is conjugated by w into another group of real Möbius transformations G'. $U \to U/G'$ is the holomorphic universal covering of S'.

The last paragraph gives the fastest way to define Teichmüller space (modulo proving the Ahlfors-Bers theorem), but does not directly display the complex analytic structure of the space.[5] This method was developed by Ahlfors to define the Teichmüller space; the derivation of the complex structure of the space of moduli then required a deep computation [3].

Bers [6] then developed a sly trick which simultaneously gave the complex structure of Teichmüller space and led to a vast new area of research in complex analysis. The whole idea can be stated so simply that we give it in detail in the next few sentences. Reflecting μ_f is the only nonholomorphic operation hence the only obstruction in getting holomorphic dependence, of the solutions to the Beltrami equation, on the functions μ_f in the unit ball in the $\mathscr{L}^\infty$ $(-1, 1)$-forms on S. Instead just extend μ_f to the lower half-plane L by the zero function!! The solution to the Beltrami equation is then a univalent function in L and its Schwarzian derivative is a holomorphic quadratic differential ω for G. w then conjugates G into a group of Möbius transformations G' with complex entries. Functions of

[5]The reflection used to extend μ_f to **C** is anti-holomorphic and destroys complex analytic dependence of $w(z)$ on μ_f.

these entries or, more traditionally, of the quadratic differentials serve as coordinates on $T(S)$. The coordinates depend on the choice of basepoint but are holomorphically related. This embedding of $T(S)$ in the quadratic differentials for G is called the *Bers embedding.*

2. The available literature. Now we can actually distinguish between the more or less available books and lecture notes on Teichmüller theory. In 1964, Bers [7] produced lecture notes at ETH which focused on the groundwork for his embedding. In 1980, I wrote a Springer Lecture Note [1] which proves a number of the basic theorems concerning the real analytic structure of Teichmüller space. It was designed to bridge the gulf between the topologists, who were getting interested in the theory at that time, and the complex analysts who had developed it. It included a description of the Fenchel-Nielsen approach to Teichmüller theory via hyperbolic structures on surfaces. Bers' Teichmüller theoretic approach to the Nielsen-Thurston classification of surface diffeomorphisms and a few minor results on group actions on Teichmüller space. In the Soviet Union, Krushkal′ and his collaborators have produced a few specialized monographs on aspects of quasiconformal mappings and Riemann surfaces (see e.g. [2]) which have many applications to Teichmüller theory but are not centrally concerned with it. A similar comment is valid for Strebel's Quadratic Differentials [18].

Until 1987, there were no other books available on the subject. Then three appeared almost simultaneously. Fred Gardiner's *Teichmüller theory and quadratic differentials* [12] and Olli Lehto's *Univalent functions and Teichmüller spaces* [15] have already been reviewed in these pages by Irwin Kra [14] and Clifford Earle [10] respectively. Gardiner's book is a monograph in the sense that it takes as a starting point the relationship between length-area arguments and quadratic differentials and develops a great deal of the theory from this viewpoint. It is an embodiment of the Teichmüller methodology. Lehto's book is mainly devoted to the complex analysis which grows out of the fact that Bers' embedding produces univalent functions in a fashion which is at once classic in style but with much new content.

Subhashis Nag's book *The complex analytic theory of Teichmüller space* is both less focused and more complete. It has quite little overlap with the other two books—then again, they have little in common with each other. For the Teichmüller theory of compact surfaces (possibly missing a finite number of points), Nag's book is the most encyclopedic, however he often needs to refer to other sources for complete proofs. The main virtue of the book lies in its complete layout of the Bers program of Teichmüller theory within the structure developed by Cliff Earle. Teichmüller's theorem is not a central player in this.[6] He shows that the complex structure of Teichmüller space which makes periods of abelian differentials holomorphic

[6]This is a mixed blessing since it does not permit treatment of Royden's characterization of the holomorphic automorphism group of $T(S)$ but this is given in detail in Gardiner's book.

is identical to that given by Bers embedding—this ties Teichmüller theory to algebraic geometry. Deformations and boundaries of Teichmüller spaces are studied as are the universal family and the universal properties of the Teichmüller space.

The book is clear and belongs on the bookshelf of anyone working in or near Teichmüller theory. It is likely to be the standard basic reference for those aspects of the Teichmüller theory of finite Riemann surfaces which are based on or proved using Bers' embedding. It is not light reading and occasionally sends the reader elsewhere for details.

Since the book does not really cover the geometry of Teichmüller space, it is clear that another book is still needed to cover such advanced topics as the Weil-Petersson and the Teichmüller-Royden geometries, the projective embedding of the compactified moduli space, the deeper theory of degenerate boundary groups, applications to various subjects such as rational billiard tables, minimal surfaces, etc.

Nag has written the book I wouldn't dare attempt and has written it well. He is to be congratulated for a major service to the community.

REFERENCES

1. W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math., vol. 820, Springer-Verlag, Berlin and New York, 1980.
2. B. Apanasov, S. Krushkal′ and N. Guseviskii, *Kleinian groups and uniformization in examples and problems*, vol. 62, Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1986.
3. L. Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces*, Analytic Functions, Princeton Univ. Press, Princeton, N.J., 1960.
4. ——, *On quasiconformal mappings*, J. d'Analyse Math. **3**(1953/1954), 1–58.
5. L. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72**(1960), 385–404.
6. L. Bers, *Correction to spaces of riemann surfaces as bounded domains*, Bull. Amer. Math. Soc. **67**(1961), 467–466.
7. ——, *On moduli of Riemann surfaces*, ETH, 1964.
8. L. Bers and H. Royden, *Holomorphic families of injections*, Acta Math. **157**(1986), 259–286.
9. L. De Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154**(1985), 137–152.
10. C. J. Earle, *Review of univalent functions and Teichmüller spaces by Olli Lehto*, Bull. Amer. Math. Soc. (N.S.) **19**(1988), 488–490.
11. C. FitzGerald and C. Pommerenke, *The De Branges theorem on univalent functions*, Trans. Amer. Math. Soc. **290**(1985), 683–690.
12. F. Gardiner, *Teichmüller theory and quadratic differentials*, J. Wiley and Sons, 1987.
13. H. Grötzsch, *Über einige extremalprobleme der konformen abildung*. I. Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Naturwiss. Kl. **80**(1928), 367–376.
14. I. Kra, *Review of Teichmüller theory and quadratic differentials by Frederick Gardiner*, Bull. Amer. Math. Soc. (N.S.) **19**(1988), 494–498.
15. O. Lehto, *Univalent functions and Teichmüller spaces*, vol. 109, Graduate Texts in Math., Springer-Verlag, Berlin and New York, 1987.
16. P. Sad, R. Manē and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16**(1983), 193–217.
17. C. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**(1938), 126–166.
18. K. Strebel, *Quadratic differentials*, Springer-Verlag, Berlin and New York, 1984.

19. D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math. **157**(1986), 243–257.

20. O. Teichmüller, *Collected Papers*, Springer-Verlag, Berlin and New York, 1982.

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Extensions and absolutes of Hausdorff spaces, by Jack R. Porter and R. Grant Woods. Springer-Verlag, New York, Berlin, Heidelberg, 1987, xiii + 856 pp., $89.00. ISBN 0-387-96212-3

The book being reviewed is designed for advanced graduate students and scholars who want up-to-date information about an important part of general topology. What this latter means is less than clear since the word "topology" is used differently from words like "algebra," "analysis," or "geometry" which also attempt to act as signposts for branches of mathematics. The *Journal of Algebra* publishes articles on groups, rings, and other parts of algebra. Most journals devoted to analysis have space for papers on real analysis, complex analysis, or differential equations, and similar statements can be made about geometry. This kind of integration contrasts with a de facto apartheid as far as topology is concerned.

There are three journals widely circulated in the United States devoted to topology. The journal *Topology* publishes articles in algebraic topology. *Topology and Applications* (formerly *The Journal of General Topology*) and *Topology Proceedings*, on the other hand, contain almost exclusively articles on general or geometric topology. This latter journal is reserved for articles presented at the annual Spring Topology Conference with an attendance of two to three hundred which only rarely attracts an algebraic topologist. There is nothing unusual about a conference attracting only specialists in a particular area, but often, in the case of topology, no qualifying adjectives seem to be used to describe its real nature. Whatever the reason, there seems to be a topological Tower of Babel.

There would seem to be three branches of the topological tree; indeed, until a few decades ago, there were about four—but what used to be called combinatorial topology is now, for the most part, subsumed under the title "graph theory" or absorbed into algebraic topology. At the center one finds geometric topology, the kind you describe to lay people who ask you to tell them what topologists do. While geometry is the study of properties of "objects" that remain invariant under rigid motions, topology is the study of properties that remain invariant under arbitrary one-one bicontinuous transformations. Geometric topologists generally confine their studies to spaces that resemble subspaces of Euclidean spaces or Hilbert spaces at least locally, and they do regard topology as a generalization of geometry. Algebraic topologists cover a lot of the same territory while putting more emphasis on (locally) Euclidean spaces, often with richer structures.