for the $\bar{\partial}$-Neumann problem. The introduction to the book gives references for these topics.

Until the publication of *Calculus on Heisenberg manifolds*, the only treatment of the Beals-Greiner calculus was their article **[BG]**. The book is very well written and it is self-contained. It makes an important topic accessible to nonexperts for the first time. The Beals-Greiner calculus should serve as a model for the development of algebras of pseudodifferential operators, with a complete calculus, suitable for handling other nonelliptic problems. In particular, it might serve as a model for a constructive method of finding parametrices for second order subelliptic differential operators which lose more than one derivative. Such operators arise naturally in a number of contexts, including the study of sums of squares of vector fields on homogeneous nilpotent Lie groups and general sums of squares of vector fields satisfying Hörmander's condition and the study of $\Box_b$ on the boundaries of weakly pseudoconvex domains. As a first step in this direction, Cummins **[C]** has developed a calculus for operators modeled on sums of squares of vector fields on three step nilpotent Lie groups.

Calculus on Heisenberg manifolds is an excellent book. In addition to giving a thorough development of the Beals-Greiner calculus, the book includes a history of symbols, kernels and $\Box_b$ in the introduction. Chapter 3 begins with a review of standard pseudodifferential operators. This review provides a very good sketch of the important features of the standard calculus (along with references for details).

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An introduction to independence for analysts, by H. G. Dales and W. H. Woodin. London Mathematical Society Lecture Notes, vol. 115, Cambridge University Press, Cambridge, New York and Melbourne, 1987, xiii + 241 pp., $29.95. ISBN 0-521-33996-0

About sixty years ago, K. Gödel proved his famous results on the incompleteness of formal theories. The fact that under very general hypotheses, a formal mathematical theory, if consistent, cannot answer to all statements expressible in its language, and in particular its own consistency, has had a profound impact on foundational issues. At the same time, it opened a new field of investigation for mathematicians: the search for independent statements, in order to understand the exact realm of the phenomenon in mathematics. And to do so, one had to invent tools for proving independence, and test these tools on known open problems. Since then, this has led to many deep results about many formal theories, including, e.g., Peano Arithmetics and its related theories, but the first and main techniques were obtained for the formal system of set theory, the so-called system ZFC of Zermelo-Fraenkel with the axiom of choice. And the book under review discusses one of the two known techniques for proving independence from ZFC, the forcing technique of P. Cohen.

A mathematical statement is independent from ZFC if it can neither be proved nor refuted from the axioms of this theory. This of course is interesting only if the theory ZFC is consistent, as otherwise it would prove any statement, as well as refute it. So the question is: given a statement φ of set theory, and assuming ZFC is consistent, how can one prove that φ is independent from ZFC?

The first approach is number-theoretic. If φ cannot be refuted in ZFC, the theory ZFC augmented with φ is consistent, i.e., does not lead to a contradiction, and this can be translated into a statement of number theory, in fact, using Matijacevic's work, one of form: Does a certain diophantine equation have no integer solution? This approach is the one used by Gödel for his incompleteness results, but has since then not produced new independent statements.

The other approach is algebraic, and deals with the notion of models of set theory; these are structures which consist of a set (which interprets the universe of all sets), and of a binary relation on it (which interprets membership), and which satisfy the axioms of ZFC, and hence also all theorems of this theory. A fundamental theorem of Gödel, the completeness theorem, asserts that a theory is consistent if and only if it has a model. So in order to prove that φ is not refutable in ZFC, it is enough to exhibit a model of ZFC in which φ is also satisfied. And as we assume ZFC itself is consistent, we may start from a model of ZFC, and try to perturb it so as to get a model of ZFC+ φ. This is the starting point of both the inner model technique, invented by Gödel in 1938 to prove that the continuum hypothesis (CH) is not refutable in ZFC, and the forcing method, created by Cohen in 1963 to prove that CH is indeed independent from it.

In the first technique, one builds a substructure of the original model, by restricting the domain without changing the membership relation. Using this idea, Gödel built his "constructible universe L," the smallest one, in a precise technical sense; it satisfies the continuum hypothesis (even generalized), the axiom of choice, even if the original model did not, as well as many other combinatorial and definability properties. Gödel's model L and similar inner models for large cardinal axioms are still one of the main domains of study in set theory, both for themselves and for the general understanding they give of the universe of sets.

In the second method, the forcing technique of Cohen, one builds a new model by extending the domain and the membership relation, in a very precise way controlled by a boolean algebra of the original model. The extension still satisfies ZFC, and the combinatorial properties of the chosen boolean algebra determine the properties of the extension. Cohen used it to show that CH cannot be proved in ZFC, and since then the forcing technique has proved to be extremely powerful in getting independence results in a great variety of situations.

This is this forcing technique which is presented to analysts in the book under review. There are already several books on this method, either text books or research ones, which cover both the basic technique and many useful applications. But these books are written for set-theorists, they presuppose an important background in logic, and their aim is to cover as many as possible of the known applications.

The book under review has a different spirit. First, it is intended for people with very few background in logic, and hence presents all basic facts, including the discussion on completeness; it also uses the "naive" approach familiar to analysts, and it gives much more detail than a technical book would do. Secondly, it aims only to present the method, and the authors have chosen to exemplify it on a specific instance of its use, an example taken from the theory of Banach algebras.

The example comes from a problem of Kaplansky. Consider the following statement, abbreviated by NDH (for No Discontinuous Homomorphism): For any compact space X, any homomorphism from the Banach algebra of continuous complex-valued functions on X into another complex Banach algebra is continuous. This statement was "refuted" independently by Dales and by Esterlé in 1976, using the Continuum Hypothesis. So in particular NDH cannot be proved in ZFC. But it cannot be refuted either: The same year, Solovay built by a forcing method a model satisfying it, using ideas of Woodin. A later simpler proof of Woodin is the one presented in the book. And en route to this consistency result, other consistency and independence results are also presented, like that of CH, and of a related combinatorial statement called Martin's axiom, which is of some interest in Analysis.

Why should an Analyst read a book on such a subject? If, as it is very probable, more and more independence phenomena are discovered and if they continue their slow diffusion through all mathematics, non set-theorists will have to face more and more often in their practice the possibility that their commonly accepted frame is not powerful enough to provide answers to their questions. Certainly, the easiest response, for the Analysts, is to accept to use as "black boxes" some already available combinatorial tools that set-theorists have shown to be consistent, like CH, the aforementioned Martin's axiom, or consequences of Gödel's axiom of constructibility. But for those who do not want to stay at this "user" level, but want to really understand how these independent results can be obtained, the book under review should be extremely fruitful.

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