

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 21, Number 1, July 1989
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 0273-0979/89 \$1.00 + \$.25 per page

Random polynomials, by A. T. Bharucha-Reid and M. Sambandham. Academic Press, Orlando, 1986, xv + 206 pp., \$59.00 cloth, \$32.95 paper. ISBN 0-12-095710-8.

In describing a physical phenomenon, mathematical equations come into the picture whose coefficients carry some physical significance. These coefficients are random variables following some probability distributions, since they are computed from experimental data or from natural observations. Thus random equations arise from many applied problems in mathematical physics, engineering and statistics. A polynomial whose coefficients are random variables is called a random polynomial. Then the coefficients are subject to random error. Although there are a lot of applications of random polynomials in various branches of science and technology, it is only recently that attempts have been made to develop the theory of random equations. The study of random algebraic polynomials was initiated by Bloch and Pólya [Proc. London Math. Soc. **33** (1932), 102–114] in 1932. Motivated by this work, the systematic study of random algebraic polynomials was initiated by Littlewood and Offord [J. London Math. Soc. **13** (1938), 288–295] in 1938. At present active research is being carried out in several countries including United States, Great Britain and India. Yet no comprehensive treatment of this subject was so far available in book form.

Using the notation of the book, we take our random polynomial in the form

$$F_n(z, \omega) = \sum_{k=0}^n a_k(\omega) z^k$$

where the coefficients $a_k(\omega)$ are random variables.

In the beginning the authors start explaining how certain concrete situations give rise to algebraic polynomials and outline a brief history of the use of probabilistic methods in the study of algebraic polynomials. The study of random algebraic polynomials gives rise to other types of random polynomials such as random trigonometric polynomials, random orthogonal polynomials and random Bernstein polynomials. The basic definitions and properties of random algebraic polynomials are introduced. The idea of random power series is introduced as a generalisation of random algebraic polynomials, and Hammersley's theorem that the zeros are Borel measurable functions of the coefficients $a_0(\omega), a_1(\omega), \dots, a_n(\omega)$ is stated. A shorter proof by Kannon is given. Continuity, separability and measurability of random algebraic polynomials are considered. It is shown that under certain conditions algebraic polynomials are martingales.

A random matrix is defined as a matrix whose elements are random variables or random functions. Random algebraic polynomials arise in the spectral theory of random matrices. It is pointed out that any matrix associated with a real (i.e., concrete) problem in the sciences, engineering,

and technology should be treated as a random matrix, since the matrix elements are often estimated, rounded off, or subject to random fluctuations due to a number of causes. Some examples of random matrices that are encountered in mathematical sciences are given.

A discussion of the number of real roots of random algebraic polynomials $\sum_{k=0}^n a_k(\omega)x^k$, $x \in R$ and its expectation is found in a separate chapter. Various results on the estimation of number of real roots of algebraic polynomials is discussed. Upper and lower bounds of the number of real roots when the coefficients are subject to different distributions are discussed. Works on this topic starting from Bloch and Pólya (1932) till today are exhaustively dealt with. The important results are for cases when the random coefficients are (i) independent random variables with normal distribution (ii) independent random variables with known mean and variance (iii) dependent normal variables (iv) independent Cauchy random variables (v) independent stable random variables. A generalisation of the Kac-Rice formula is presented, and estimates of the number of real zeros are given with the help of the formula. Some results on the expected number of real zeros when the coefficients of the random algebraic polynomial are complex-valued random variables are also presented.

The number and expected number of real zeros of random trigonometric polynomials and random hyperbolic polynomials is considered and certain numerical results are derived.

A discussion of the variance of the number of real zeros of random algebraic polynomials is made. It includes the cases when (i) random coefficients are independent Gaussian random variables with mean zero and variance one (ii) the random coefficients assume the values ± 1 with equal probability. The main theorem deals with asymptotic estimates of the variance of the number of real zeros of the random algebraic polynomial when the coefficients are dependent standard Gaussian random variables with given joint density function. The theoretical estimates are tested with computer-generated numerical results.

A separate chapter deals with the determination of the probability distributions of the zeros of a random polynomial when the probability distributions of random coefficients are given. The problem of determining the distribution of the solution of the random linear equation and the random quadratic equation is also considered.

The last chapter highlights the limiting behaviour of the number of zeros of random algebraic and random trigonometric polynomials. A few results on the averaging problem for the zeros of random algebraic polynomials and random companion matrices are presented here.

Almost all available results on random polynomials are presented in this book. The book is the first of its kind in presenting a rigorous treatment on the subject. For everybody who works in the field the book provides all information.