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*Operator theory and arithmetic in* $H^\infty$, by Hari Bercovici. Mathematical Surveys and Monographs, No. 26, American Mathematical Society, Providence, R.I., 1988, xii + 275 pp., $67.00. ISBN 0-8218-1528-8

Linear algebra is supposed to be in every mathematician's toolbox. Somewhere in our undergraduate instruction, we learned that every finite matrix is similar to a uniquely determined Jordan form (resp. rational form), which reflects all its similarity-invariant properties. For operator theorists whose working environment is bounded linear operators on complex Hilbert spaces, a suitable generalization to this context is very much desired. While the problem of developing a canonical model for general operators may be beyond the reach of present day operator theory, practitioners in this field try another approach by isolating classes of operators for which such models can be obtained. One such class is that of *algebraic operators*, that is, operators $T$ for which $p(T) = 0$ for some nonzero polynomial $p$. That every finite matrix is algebraic is a consequence of the Cayley-Hamilton theorem. The Jordan canonical form for finite matrices can be generalized to this class: every algebraic operator is quasisimilar to a unique "Jordan operator" [7]. Here the weaker notion of quasisimilarity replaces that of similarity in finite dimensions. Recall that two operators $T$ and $S$ are *quasisimilar* if there are operators $X$ and $Y$ which are injective and have dense range such that $TX = XS$ and $SY = YT$. To a large degree, these models reflect faithfully properties of the original operators. Recently, this was extended by K. R. Davidson and D. A. Herrero **[2]** to the class of *bitriangular operators*; these are operators $T$ for which both $T$ and $T^*$ have upper triangular matrices with respect to some (possibly different) orthonormal bases of the underlying space. Moreover, they showed that such operators form the largest class of operators for which a "Jordan model" can be constructed.

The monograph under review is concerned with the structure and particularly the Jordan canonical form of another class of operators—that of $C_0$ contractions. A contraction $T$ ($\|T\| \leq 1$) is of *class* $C_0$ if $T$ is completely nonunitary and $\phi(T) = 0$ for some nonzero $\phi$ in the Hardy algebra $H^\infty$ of bounded analytic functions on the unit disc. (Here the completely nonunitary assumption is required to guarantee that the Sz.-Nagy-Foiaş functional calculus $\phi(T)$ is meaningful.) This class of operators was discovered by B. Sz.-Nagy and C. Foiaş in 1964 **[4]** in their work on the functional model for contractions on Hilbert space. In the more than two decades since then, a whole theory of these operators has been developed. Besides its two founders, the present author is a major contributor. Note that the theory of $C_0$ contractions contains properly that of algebraic operators since if $T$ is algebraic then $\alpha T$ is of class $C_0$ for $0 < \alpha < 1/\|T\|$. Their structure resembles closely that of operators on finite-dimensional spaces. The whole theory is rich in interplay between operator theoretic properties on the one hand and arithmetic properties of $H^\infty$ on the other. At its present stage of development, the class $C_0$ is, as put aptly by the author, probably the best understood class of operators on Hilbert space other than the normal ones.

Here are some highlights of the theory. If $T$ is a $C_0$ contraction, then $T$ has a minimal inner annihilating function in $H^\infty$ which we denote by $m_T$. Just as the eigenvalues of a finite matrix can be computed from its minimal polynomial, the spectrum and the point spectrum of $T$ can be recovered from $m_T$. The role of Jordan cells in the Jordan form of $T$ is played by the operator $S(\theta)$, where $\theta$ is an inner function in $H^\infty$ and $S(\theta)$ is the operator acting on $H^2 \ominus \theta H^2$ by

$$S(\theta)h = P(e^{it}h(e^{it})),$$

$P$ being the orthogonal projection from $H^2$ onto $H^2 \ominus \theta H^2$. If $T$ acts on a separable Hilbert space, then $T$ is quasisimilar to a unique *Jordan operator* of the form

$$\bigoplus_{n\geq 1} S(\theta_n),$$

where $\theta_n$'s are inner functions satisfying $\theta_m \mid \theta_n$ for $m > n$. (If $T$ is nonseparably acting, then some complications involving ordinal numbers may arise.) Hence the "Jordan model" here is more like the rational form for finite matrices. In particular, this classifies $C_0$ contractions into quasisimilarity classes. Note that in the Jordan model, $\theta_1$ is necessarily the minimal function of $T$. Certain properties of $T$ involving lattices of invariant subspaces and operator algebras associated with $T$ can be related to the corresponding ones of its Jordan model. One example is the characterization of reflexive $C_0$ contractions: $T$ is reflexive if the only if $S(\theta_1/\theta_2)$ is ($T$ is *reflexive* if the only operators leaving invariant every invariant subspace of $T$ are those in Alg $T$, the weakly closed algebra generated by $T$ and $I$). This is the counterpart of a corresponding result for finite matrices. All the above can be found in the first four chapters of the book.

From Chapter 5 on, the Jordan model is related to the Sz.-Nagy-Foiaş functional model of $T$. The main concern here is to calculate the inner functions $\theta_n$ in the Jordan model using the functional model and the characteristic function of $T$. One major result is a diagonalization theorem for matrices over $H^\infty$, analogous to the diagonalization theorem for finite matrices over a principal ideal domain. The fact that $H^\infty$ is not such a domain constitutes the main difficulty. The diagonalization yields the invariant factors of matrices over $H^\infty$ which can then be used to obtain the Jordan model of a $C_0$ contraction just as the rational form of a finite matrix $T$ can be calculated from the invariant factors of its characteristic matrix $\lambda I - T$, $\lambda \in \mathbf{C}$. Another high point is the defining of the class of $C_0$*-Fredholm operators* with respect to each $C_0$ contraction $T$. This class generalizes the classical Fredholm operators: the latter corresponds to the case $T = 0$. In order for the definition to make sense, generalized notions of "dimension" and "index" have to be defined in terms of $T$. The theory has all the ingredients of the classical Fredholm theory including the index additivity property.

The monograph gives a comprehensive exposition of this beautiful theory. Most of the results are in book form for the first time. It is well written and clearly presented. The author has tried to make it as self-contained as possible by including some introduction to the dilation theory and functional model of contractions. Each chapter starts with a lucid summary of what is to come and each section ends with a set of well-chosen exercises. It can serve as a graduate textbook after a standard functional analysis course, a book for seminar topics or a monograph for research reference. It is also a stepping stone toward a better understanding of Sz.-Nagy and Foiaş' contraction theory as presented in [**5**].

The reviewer noticed relatively few misprints. Some discrepancies of the terminology do occur: antilinear map (p. 37) is the same as conjugate linear map (p. 64); $\{T\}''$ (p. 74) has been called double commutant (p. 182) and bicommutant (p. 227). To the references he would suggest to add [**1**, **3**, and **6**].

In summary, the author has done an outstanding job presenting a part of operator theory which, because of its intrinsic interest and potential applications to systems theory, deserves more attention among practitioners in this field.

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Pei Yuan Wu
National Chiao Tung University

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*Stratified Morse theory*, by M. Goresky and R. MacPherson. Springer-Verlag, Berlin, Heidelberg, New York, 1988, xiv + 272 pp., $75.00. ISBN 3-540-17300-5

An important tool in the investigation of the topology of a differentiable manifold $M$ is the classical Morse theory. Given a Morse function $\varphi$, i.e., a proper differentiable function $\varphi\colon M \to \mathbf{R}$, bounded from below and