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Introduction to superanalysis, by Felix Alexandrovich Berezin. Edited by A. A. Kirillov. Translated by J. Niederle and R. Kotecký. Mathematical Physics and Applied Mathematics, vol. 9, D. Reidel Publishing Company, Dordrecht, 1987, xii + 424 pp., $109.00. ISBN 90-277-1668-4

Superanalysis, by F. A. Berezin, is a posthumous work, combining several of Berezin's preprints on Lie supergroups and superalgebras with fragments of a textbook on supermanifolds. Included also is an appendix by V. I. Ogievetsky on supersymmetry and supergravity.

At its foundational level, superanalysis is a simple generalization of commutative algebraic geometry. The basic notion is that of a $\mathbf{Z}_2$-graded vector space, a vector space V with a decomposition into two subspaces: $V = V_0 \oplus V_1$. For $v \in V_i$, i is called the *parity* of v, and is denoted by $|v|$. If V and W are $\mathbf{Z}_2$-graded vector spaces, then there is a supertwist map

$$V \otimes W \to W \otimes V$$
$$v \otimes w \to (-1)^{|w||v|} w \otimes v.$$

A $\mathbf{Z}_2$-graded algebra A is *supercommutative* if it satisfies the *usual* diagram for commutativity

$$\begin{array}{ccc} A \otimes A & \xrightarrow{T} & A \otimes A \\ \text{mult} & & \text{mult} \\ & A & \end{array}$$

where T is now the *supertwist*.

Thus supercommutative algebras, being in a sense commutative, inherit virtually unaltered the elementary theory of their commutative counterparts. The subject of superanalysis therefore organizes itself around the following themes:

1. Any superanalytic object will ultimately bear a strong resemblance to its classical analogue.

2. The understanding the first theme often requires more than a superficial understanding of the classical case, and this in itself makes the study of superanalysis worthwhile.

Berezin's book seems to resist these guiding principles. It conveys rather the impression that with the introduction of anticommuting coordinates, a new world of "supermathematics" appears.

The introductory chapter lays out the definitions of supermanifold, Lie superalgebra, integral and differential calculus on superspace. These definitions are repeated and expanded in later chapters.

Chapter 1 is entitled *Grassmann algebra*. By Grassmann algebra we mean an exterior algebra, $\Lambda_A(M)$, where A is a commutative algebra and M is a free A-module. If A is the algebra of smooth functions on a domain $U \subset \mathbf{R}^n$, and if a set of generators $\xi^1, \dots, \xi^m$ is given for M, then

Berezin denotes $\Lambda_A(M)$ by $\Lambda(U)$. $\Lambda(U)$ is made into a supercommutative algebra by declaring the elements of A to be even and the generators of M to be odd. Given any $\mathbf{Z}_2$-graded algebra B, a *graded derivation*, or *superderivation* of B is a linear map $D\colon B \to B$ satisfying

$$D(fg) = D(f)g + (-1)^{|D||f|} f D(g). \tag{1}$$

The Lie superalgebra of graded derivations of $\Lambda(U)$ is freely generated by $\partial/\partial x^1, \ldots, \partial/\partial^n, \partial/\partial\xi^1, \ldots, \partial/\partial\xi^m$. For a given number of odd generators, m, denote the sheaf $U \supseteq V \to \Lambda(V)$ by $U^{n|m}$. This is a sheaf of supercommutative algebras. Such a sheaf is called a *superspace*. The essential point is

PROPOSITION 1 **[M]**. *Let U and V be connected open subsets of $\mathbf{R}^n$. Then there are natural bijections between the following sets*

(1) Hom$(U^{n|m}, V^{n|m})$ *in the category of superspaces.*
(2) Hom$(\Gamma(U, U^{n|m}), \Gamma(V, V^{n|m}))$ *in the category of supercommutative algebras over* $\mathbf{R}$.
(3) *Tuples*

$$\begin{gathered}(y^1(x,\xi), \ldots, y^n(x,\xi),\ \eta^1(x,\xi), \ldots, \eta^m(x,\xi)) \\ \in (\Gamma(U, U_0^{n|m}))^n \oplus (\Gamma(U, U_1^{n|m}))^m\end{gathered}$$

such that for all $x \in U$, $y(x,0) \in V$. The morphism of superspaces defined by (y,η) is a local isomorphism in a neighborhood of a point $x \in U$ if and only if the matrix of partials,

$$\begin{pmatrix} \dfrac{\partial y^i}{\partial x^j} & \dfrac{\partial \eta^a}{\partial x^j} \\ \dfrac{\partial y^i}{\partial \xi^b} & \dfrac{\partial \eta^a}{\partial \xi^b} \end{pmatrix}, \tag{2}$$

is invertible at x.

From Proposition 1 it follows that there are categories of smooth, analytic or algebraic supermanifolds, on which one may develop a differential calculus in analogy with differential calculus on purely even manifolds. For instance, a C^∞ *supermanifold* is a ringed space $(X, \mathscr{A})$ which is locally isomorphic to $U^{n|m}$, as a sheaf of supercommutative algebras over $\mathbf{R}$. *Superanalysis* does not pursue this simple line directly, but presents instead a lengthy discussion of such aspects of the Grassmann algebra as its *ungraded* automorphism group, and the various sorts of involutions one can introduce.

Next follows a chapter entitled *Superanalysis*, where integration and differentiation are discussed again. Integration on supermanifolds was Berezin's invention, and it is the first aspect of supergeometry in which the generalization from the purely even case is not obvious. Given a compactly supported superfunction, $f \in \Lambda(U)$, set $f = \sum_\mu f_\mu(x)\xi^\mu$, μ being a multi-index of 0's and 1's, and define

$$\int_U f(x,\xi)\,d(x,\xi) = \int f_{(1,\ldots,1)}(x)\,dx. \tag{3}$$

Given new coordinates, (y, η), consider the Jacobian matrix, (2). Set

$$\mathscr{A} = \frac{\partial x}{\partial y} \qquad \mathscr{B} = \frac{\partial x}{\partial \eta}$$
(4)
$$\mathscr{C} = \frac{\partial \xi}{\partial y} \qquad \mathscr{D} = \frac{\partial \xi}{\partial \eta}.$$

Associated to matrix (2) is its *berezinian*,

(5) $$\operatorname{Ber}\begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} = \operatorname{Det}(\mathscr{A} - \mathscr{B}\mathscr{D}^{-1}\mathscr{C}) \operatorname{Det}(\mathscr{D}^{-1}).$$

Then $d(x, \xi)$ is assumed to transform by the berezinian. Chapter 2 gives a nice proof of the multiplicativity of Ber, based on the transformation properties of the integral.

Chapter 3, *Linear algebra in* $\mathbf{Z}_2$*-graded spaces*, offers a second proof of the multiplicativity of the superdeterminant. It is clearly wrong as stated, presumably because of some huge typographical errors. Indeed, misprints abound throughout the text.

Chapter 4, *Supermanifolds in general*, brings with it a dramatic increase in organization and clarity. This chapter, which was rewritten by V. P. Palomodov, stands nicely on its own as an introduction to the notions of sheaf, ringed space, and supermanifold. The main results of this chapter have to do with obstructions to the existence of a projection from a supermanifold onto its body. Let $(X, \mathscr{A})$ be a C^∞ supermanifold. Since $(X, \mathscr{A})$ is locally isomorphic to $\Lambda(U)$, one has a sheaf of nilpotents $\mathscr{N} \subset \mathscr{A}$, and an exact sequence

(6) $$0 \to \mathscr{N} \to \mathscr{A} \to C^\infty \to 0.$$

By "projection," we mean a splitting of sequence (6). In fact, sequence (6) always splits, but if one considers instead complex supermanifolds, then there are many natural nonsplit examples. Proofs of both assertions may be found in the text. See also [**B**, **Ga**, **Gr**, and **M**].

The remainder of the book is about Lie superalgebras and Lie supergroups. Chapter 5, *Lie superalgebras* is the end of part I, which is the part that was to be Berezin's textbook. Part II, based on Berezin's preprints, reflected his interest in extending his work on Laplace-Casimir operators to the supercase. The focus throughout part II is on the special cases of supergroups $U(p, q)$ and $C(m, n)$. The classical theory meets with several difficulties in its passage to the supercase, and most of the rest of the book is spent in overcoming these difficulties in the special cases cited above. Along the way, further machinery is developed, notably the notion of Lie supergroup, and the invariant integral. A supergroup may be defined as a group object in the category of supermanifolds. Alternatively, one may imitate the functor associating to a Lie group the Hopf algebra structure on its space of functions, and define a supergroup as a kind of super Hopf algebra. The latter approach is chosen here, and may also be found in [**K**].

The main difficulty of the Laplace-Casimir theory in the supercase is that, in contrast to the classical case, a Weyl invariant polynomial on the

Cartan subalgebra need not give rise to a Laplace operator on the supergroup. The author shows, however, that under suitable hypotheses, satisfied for $U(p,q)$ and $C(m,n)$, one can recover the classical result by considering instead *rational* Weyl invariant functions on the Cartan and the *field of fractions* of the algebra of Laplace-Casimir operators. Unfortunately, this interesting idea is tossed of rather lightly, leaving its meaning unclear (at least to the reviewer).

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Gauge field theory and complex geometry, by Yuri I. Manin. Translated from the Russian by N. Koblitz and J. R. King, Springer-Verlag, Berlin, Heidelberg, 1988, x + 295 pp., $80.00. ISBN 0387-18275-6

Few areas of twentieth century scientific thought have provoked more puzzlement, outrage, and disbelief among the general populace than that part of modern physics which asserts that the true geometry of the natural world is profoundly different from the "common sense" geometry canonized by Euclid. The definitive revolution in this area was, of course, that wrought by Einstein, Minkowski, and their contemporaries, who discovered a new geometry, not of space but rather of space-time. This change of perspective had, in retrospect, been waiting to happen ever since Maxwell wrote down his field equations for electromagnetism; it was not new physics, but rather the casting of the symmetries of the old physics in a geometrical guise, that brought the new geometry into existence.

In the last decades, a number of new geometric ideas have entered the arena of theoretical physics, often with lasting repercussions for mathematics. The present book deals with three families of such ideas: those of nonabelian gauge-field theory, of twistor theory, and of supersymmetry. The main thrust of the work centers on an exegesis of a paper in which Ed