

## PLEATING COORDINATES FOR THE TEICHMÜLLER SPACE OF A PUNCTURED TORUS

LINDA KEEN AND CAROLINE SERIES

**ABSTRACT.** We construct new coordinates for the Teichmüller space  $\text{Teich}$  of a punctured torus into  $\mathbf{R} \times \mathbf{R}^+$ . The coordinates depend on the representation of  $\text{Teich}$  as a space of marked Kleinian groups  $G_\mu$  that depend holomorphically on a parameter  $\mu$  varying in a simply connected domain in  $\mathbf{C}$ . They describe the geometry of the hyperbolic manifold  $\mathbf{H}^3/G_\mu$ ; they reflect exactly the visual patterns one sees in the limit sets of the groups  $G_\mu$ ; and they are directly computable from the generators of  $G_\mu$ .

### 1. INTRODUCTION

In this paper we announce results that will appear in [4], in which we construct a new embedding in  $\mathbf{R} \times \mathbf{R}^+$  of the Teichmüller space  $T_{1,1}$  of a punctured torus. The pullbacks of the natural coordinates in  $\mathbf{R} \times \mathbf{R}^+$  have simple geometric interpretations when  $T_{1,1}$  is realized as a subset of  $\mathbf{C}$  using the Maskit embedding. In the Maskit embedding, points of  $T_{1,1}$  correspond to marked Kleinian groups  $\{G_\mu\}$  that depend holomorphically on a parameter  $\mu$  that varies in a simply connected domain  $\mathcal{M}$  in  $\mathbf{C}$ . Figure 1 (see p. 142), drawn by David Wright, shows the domain  $\mathcal{M}$ . The 'coordinate grid' in  $\mathcal{M}$  is the preimage under our embedding of the horizontal and vertical lines in  $\mathbf{R} \times \mathbf{R}^+$ .

Our coordinates, which we call *pleating coordinates*, have the following virtues: they relate directly to the geometry of the hyperbolic manifold  $\mathbf{H}^3/G_\mu$ ; they reflect exactly the visual patterns one sees in the limit sets of the groups  $G_\mu$ ; and they are directly computable from the generators of  $G_\mu$ . The definition of the Maskit embedding prescribes that the regular set of each group  $G_\mu$  should contain precisely one invariant component  $\Omega_0(G_\mu)$ . The Riemann surface  $\Omega_0(G_\mu)/G_\mu$  is a punctured torus, and this defines the correspondence between  $G_\mu$  and a point in  $\mathcal{M}$ . Ideally, one would like to relate the geometry of the surface  $\Omega_0(G_\mu)/G_\mu$  directly to the parameter  $\mu$ . This seems to be a very hard problem. However, it is possible to determine the relationship of  $\mu$  to the geometry of the hyperbolic manifold  $\mathbf{H}^3/G_\mu$ . This is the basic idea of [4].

The boundary of the convex hull of the limit set of a Kleinian group acting on  $\mathbf{H}^3$  carries an intrinsic hyperbolic metric and is a union of pleated surfaces in the sense of Thurston (see [9,1]). (Roughly speaking, a pleated surface is a hyperbolic surface in a hyperbolic 3-manifold that is bent or pleated along some geodesic lamination called its *pleating locus*.) We use the term *pleating coordinates* because our embedding reflects the geometry of this pleating.

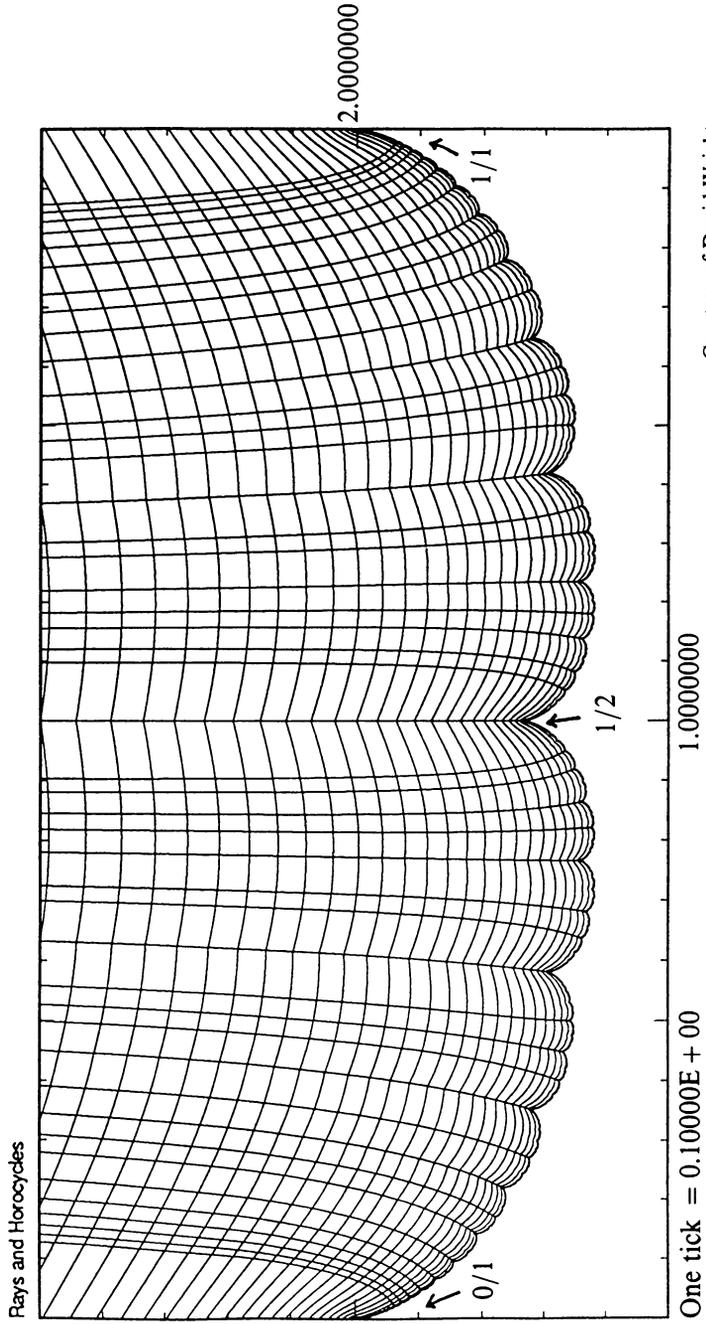
Since there is a natural bijective correspondence between the connected components of the regular set of  $G_\mu$  and those of its convex hull boundary, exactly

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Received by the editors March 29, 1991 and, in revised form, June 5, 1991.

1991 *Mathematics Subject Classification*. Primary 30F40, 32G14.

Research partially supported by NSF grant DMS-8902881.



Courtesy of David Wright

FIGURE 1. The Maskit embedding with pleating coordinates.

one of these boundary components,  $\partial\mathcal{E}_0$  say, is invariant under the action of  $G_\mu$ . The two quotients  $\partial\mathcal{E}_0/G_\mu = \widehat{\mathcal{F}}_\mu$  and  $\Omega_0(G_\mu)/G_\mu$  are topologically, but not conformally, the same (see [1,3]). Thus the pleated surface  $\widehat{\mathcal{F}}_\mu$  is topologically a punctured torus (hereafter denoted by  $S$ ). The parameters we use in our embedding reflect the geometry, not of  $\Omega_0(G_\mu)/G_\mu$ , but of  $\widehat{\mathcal{F}}_\mu$ . In the rest of this paper, we shall see how the pleating coordinates describe  $\widehat{\mathcal{F}}_\mu$  in terms of its pleating locus  $\text{pl}(\mu)$ .

## 2. PLEATING RAYS

The ‘vertical’ lines of the coordinate grid in Figure 1 are the locus of points in  $\mathcal{M}$  along which  $\text{pl}(\mu)$  is a particular fixed geodesic lamination on  $S$ . We call these lines *pleating rays*. They may be thought of as ‘internal rays’ in  $\mathcal{M}$  since they play a role analogous to that of the ‘external rays’ of the Mandelbrot set in the study of the dynamics of quadratic polynomials.

Define a *projective measured lamination* to be a geodesic lamination together with a projective class of transverse measures. The set of projective measured laminations on  $S$  is naturally identified with  $\mathbf{R} = \mathbf{R} \cup \{\infty\}$  (see [8]). Because the pleating locus always carries a natural transverse measure, the *bending measure* (see [9,1]), we obtain for each  $\mu$  a projective measured lamination on  $S$ , also denoted  $\text{pl}(\mu)$ . For each  $\lambda \in \mathbf{R}$  ( $\lambda \neq \infty$ ), there is a unique nonempty pleating ray

$$\mathcal{P}_\lambda = \{\mu \in \mathcal{M} : \text{pl}(\mu) = \lambda\}.$$

We show that the ray  $\mathcal{P}_\lambda$  is asymptotic to the line  $\Re\mu = 2\lambda$  as  $\Im\mu \rightarrow \infty$ .

As is well known [8], the simple closed curves on  $S$  correspond exactly to  $\mathbf{Q} \cup \{\infty\}$ . More precisely, for each rational  $p/q$ , there is a unique free homotopy class  $[\gamma_{p/q}] \in \pi_1(S)$  for which the corresponding geodesic on the unpunctured torus is in the  $(p, q)$ -homology class, and for which the geodesic  $\gamma_{p/q}$  on the punctured torus is simple. We call the sets  $\mathcal{P}_{p/q}$ , *rational pleating rays*.

The group  $G_\mu$  is an embedding of  $\pi_1(S)$  in  $SL(2, \mathbf{C})$ , and the free homotopy class of  $\gamma_{p/q}$  corresponds to a conjugacy class of  $G_\mu$  under this embedding. Choose  $g_{p/q}(\mu)$  in this conjugacy class.

**Theorem 1.** *The rational ray  $\mathcal{P}_{p/q}$  is the unique branch of the locus  $\{\mu \in \mathbf{C} : \text{Tr } g_{p/q}(\mu) > 2\}$  that is asymptotically vertical as  $\Im\mu \rightarrow \infty$ . This branch contains no singularities, and it intersects  $\partial\mathcal{M}$  in a unique point. At this point,  $\text{Tr } g_{p/q}(\mu) = 2$ .*

It is not hard to show that the rational ray  $\mathcal{P}_{p/q}$  must be contained in the locus where  $\text{Tr } g_{p/q}(\mu)$  is real. The asymptotic behavior of the trace polynomials is a straightforward consequence of the trace identities, as described in more detail below. What is much more interesting is that the pleating ray is *precisely* that branch of the real locus defined above. The key point in proving this is to show that along  $\mathcal{P}_{p/q}$ , the invariant component  $\Omega_0(G_\mu)$  is a *circle chain*, that is, a union of overlapping circles that fit together in a manner reflecting the continued fraction expansion of  $p/q$ . There are two main points: for sufficiently large  $c > 0$ , the ray  $\mathcal{P}_{p/q}$  intersects the line  $\Im\mu = c$  in a unique point, and, circle chains persist under continuous deformations of the group  $G_\mu$  along  $\mathcal{P}_{p/q}$ . The endpoint of the pleating ray represents a cusp group in which

the element  $g_{p/q}(\mu)$  has become parabolic. In [2], we prove that there are no other cusp groups corresponding to  $g_{p/q}(\mu)$  in  $\partial\mathcal{M}$ .

The circle chain patterns are visually apparent in computer pictures of the limit sets of groups on the pleating ray and near to  $\partial\mathcal{M}$ . When  $\mu$  reaches the endpoint of the pleating ray, the overlapping circles become tangent. Such chains of tangent circles were discovered by David Wright in the course of a computer investigation of  $\partial\mathcal{M}$ , and our interest in them was the starting point of the present work.

Wright obtained his striking pictures of  $\partial\mathcal{M}$  by using an inductive procedure related to the continued fraction expansion of  $p/q$  to canonically choose a particular word  $W_{p/q} \in G_\mu$  in the conjugacy class of the image of  $[\gamma(p/q)]$ . He computed  $\text{Tr } W_{p/q}$  as a polynomial in  $\mu$  by means of the trace identities; and, by using his enumeration scheme to give a systematic choice of initial point, he used Newton's method to find, for each  $p/q$ , a particular root of the equation  $\text{Tr } W_{p/q}(\mu) = 2$ . These solutions form the boundary curve in Figure 1.

### 3. PLEATING LENGTH

As  $\mu$  moves down each rational ray, the length in  $\mathbf{H}^3$  of the pleating locus  $\text{pl}(\mu)$  provides a natural parameter. This parameter, however, is not continuous as  $\mu$  moves across rays. In fact, if  $\mu_n \in \mathcal{P}_{p_n/q_n}$  converges to  $\mu \in \mathcal{P}_\lambda$ , where  $\lambda$  is not rational, the hyperbolic lengths of the pleating loci  $\text{pl}(\mu_n)$  always approach infinity. However, it is possible to define a global length parameter that is continuous as  $\mu$  moves across rays by making a special choice of transverse measure, which we call the *pleating measure*, for each projective measured lamination  $\mu$  on  $S$ . We define the *pleating length*  $PL(\mu)$  of  $G_\mu$  to be the length of  $\text{pl}(\mu)$  with respect to the pleating measure of  $\text{pl}(\mu)$ . On a rational ray  $\mathcal{P}_{p/q}$ , the pleating length of  $G_\mu$  turns out to be the hyperbolic length of  $\gamma_{p/q}(\mu)$  divided by the intersection number of  $\gamma_{p/q}$  with the fixed curve  $\gamma_\infty$ . The pleating length gives a natural parameterization of the pleating rays, and the horizontal lines of the coordinate grid in Figure 1 are lines of constant pleating length.

To prove continuity properties of the pleating measure and pleating length, we use the continuous dependence on  $\mu$  of the hyperbolic structure of the convex hull boundary, the pleating locus, and the bending measure. These facts are also needed in the proof of Theorem 1. We prove all of these results in a more general setting in [3].

### 4. THE COORDINATES

It is apparent from Figure 1 that the partial foliation of  $\mathcal{M}$  by the rational rays should extend to a foliation by the real rays  $\mathcal{P}_\lambda$ ,  $\lambda \in \mathbf{R}$ . To show that it does, we characterize the irrational pleating rays as the real loci of a family of holomorphic functions. The *complex translation length* of a loxodromic element  $g \in SL(2, \mathbf{C})$  is defined as  $2 \text{arccosh}(\text{Tr } g)/2$  (see [9]). It follows from Theorem 1 that on the rational ray  $\mathcal{P}_{p/q}$ , the polynomial  $\text{Tr } g_{p/q}(\mu)$  is real valued and, hence, that the complex translation length is real. Since  $\mathcal{P}_{p/q}$  is connected, we can choose a well-defined branch of the complex translation length of  $g_{p/q}(\mu)$  by specifying that it be real on  $\mathcal{P}_{p/q}$ . We show that the family of functions  $\{L_{p/q} = 1/q \text{arccosh}(\text{Tr } g_{p/q}(\mu))\}_{p/q \in \mathbf{Q}}$  is normal in  $\mathcal{M}$  and that on  $\mathcal{P}_{p/q}$ , the function  $L_{p/q}(\mu)$  coincides with the pleating length  $PL(\mu)$ . Taking limits in  $\mathcal{O}(\mathcal{M})$ , the space of analytic functions on  $\mathcal{M}$  with the topology

of uniform convergence on compact subsets, we prove

**Theorem 2.** *The family  $\{L_{p/q}\}$  extends to a family  $\{L_\lambda\}_{\lambda \in \mathbf{R}}$  of complex analytic functions defined on  $\mathcal{M}$ , such that the function from  $\mathcal{M}$  to  $\mathbf{R}$ , given by  $\mu \mapsto PL(\mu)$ , and the function from  $\mathbf{R}$  to  $\mathcal{O}(\mathcal{M})$ , given by  $\lambda \mapsto L_\lambda$ , are both continuous and such that the function  $L_\lambda$  is real valued on  $\mathcal{P}_\lambda$ .*

That the real rays are a codimension one foliation of  $\mathcal{M}$  follows from

**Theorem 3.** *The real pleating ray  $\mathcal{P}_\lambda$  is a connected component of the real locus of  $L_\lambda$  in  $\mathcal{M}$ . This component contains no singularities and is asymptotic to  $\Re\mu = 2\lambda$  as  $\Im\mu \rightarrow \infty$ .*

Our main theorem is

**Theorem 4.** *The map from  $\mathcal{M}$  to  $\mathbf{R} \times \mathbf{R}^+$  defined by  $\mu \mapsto (\text{pl}(\mu), PL(\mu))$  is a homeomorphism onto its image.*

The fact that the map described in Theorem 4 is surjective will be proved elsewhere.

In a future paper, we expect to use the methods described here to give a complete description of  $\partial\mathcal{M}$  and of the approach to  $\partial\mathcal{M}$  along the internal rays. In particular, we hope to give proofs of McMullen's theorems [6,7], conjectured by Bers, that the cusp groups are dense in  $\partial\mathcal{M}$  and that  $\partial\mathcal{M}$  is a Jordan curve. Although our work here relates to the punctured torus, most of the techniques we have developed apply more generally. We plan to extend our analysis to any union of surfaces of finite topological type. David Wright has already produced computer pictures of the analogous coordinatization for the (one complex dimensional) Riley slice of Schottky space, and the discussion in [4] goes over to that situation (see [5]).

#### ACKNOWLEDGMENTS

We wish to express our thanks to a number of people. David Wright introduced us to this problem and has generously allowed us to use his computer pictures. Curt McMullen has graciously shared his ideas and work with us. We also want to thank Jonathan Brezin, David Epstein, Michael Handel, Steve Kerckhoff, Paddy Patterson, and Bill Thurston for many helpful conversations throughout the course of this work. Finally, we would like to acknowledge the support of the NSF, the SERC in the United Kingdom, the Danish Technical University, and the IMS at SUNY at Stonybrook.

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