General orthogonal polynomials, by Herbert Stahl and Vilmos Totik. Encyclopaedia of Mathematics and its Applications, vol. 43, Cambridge University Press, Cambridge, 1992, 250 pp., $59.95. ISBN 0-521-41534-9

If the nineteenth century was a period in which orthogonal polynomials were treated as special functions, for which all manner of identities had to be established, then the twentieth century has been a period in which they have been viewed as objects worthy of analysis, especially their asymptotic behaviour as the degree tends to infinity. The research monograph under review presents a polished part of the asymptotic theory.

Recall that if μ is a nonnegative Borel measure with support supp(μ) in the complex plane $\mathbb{C}$ containing infinitely many points, and if all *moments*

$$\mu_j := \int t^j \, d\mu(t), \qquad j = 0, 1, 2, \ldots,$$

are finite, then there exists a unique sequence of *orthonormal polynomials*

$$p_n(\mu; z) = \gamma_n(\mu) z^n + \cdots, \qquad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying the orthogonality relations

$$\int p_n(\mu; z) \overline{p_m(\mu; z)} \, d\mu(z) = \delta_{mn}.$$

Clearly this can be formulated as orthogonality within an inner product space, and the Gram-Schmidt process may be used to generate $\{p_n(\mu; z)\}_{n=0}^{\infty}$.

Orthogonal polynomials have a myriad of applications, namely, in numerical analysis, approximation theory, combinatorics, special functions, statistical physics, signal processing and speech synthesis, ... , and even design of astronomical observatories. Interest in them has continued to grow, and there is at least one annual conference devoted to their theory and applications.

Yet in the last century, the term "orthogonal polynomial" was hardly used, and usually they masqueraded in some other form. For example, in his famous posthumously published memoir [7], Stieltjes introduced Riemann-Stieltjes integration and orthogonal polynomials with respect to fairly general measures only to help describe the values of certain continued fractions.

Orthogonal polynomials seemed to have emerged as objects worthy of independent study in the 1920s. Still, in his papers (see [8]) that gave birth to outer (or Szegö) functions and H_2 spaces, Szegö used Toeplitz forms, not orthogonal polynomials, in the title, despite the fact that these papers were instrumental in developing an asymptotic theory for orthogonal polynomials for measures μ supported on the unit circle.

It is somewhat surprising to discover that even for classical Jacobi polynomials [orthogonal for $\mu'(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ on $(-1, 1)$] or Hermite polynomials [orthogonal for $\mu'(x) = \exp(-x^2)$ on $\mathbb{R}$], much of the analysis—bounds, asymptotic behaviour, and so on—was only carried out this century, and then by the "greats" of the subject: Szegö, Bernstein, Faber, and others like Einar Hille.

Perhaps the most widely used polynomials are the Chebyshev polynomials $T_n(x) := \cos(n \arccos x)$, $x \in (-1, 1)$, $n \geq 1$, orthogonal for the weight $\mu'(x) = 1/\sqrt{1 - x^2}$ on $[-1, 1]$. They already illustrate very many of the properties characteristic of general orthogonal polynomials; their oscillatory behaviour on the interval of orthogonality $(-1, 1)$ is obvious from their definition. Off the interval of orthogonality, they exhibit the characteristic geometric growth.

From the identity

$$T_n(\tfrac{1}{2}(w + w^{-1})) = \tfrac{1}{2}(w^n + w^{-n}), \qquad w \in \mathbb{C} \backslash \{0\},$$

and

$$z = \tfrac{1}{2}(w + w^{-1}) \Leftrightarrow w = \psi(z) = z + \sqrt{z^2 - 1}, \qquad z \in \mathbb{C} \backslash [-1, 1], \ |w| > 1,$$

where the branch of the square root is chosen so that $\sqrt{}$ is positive in $(0, \infty)$, it is easily seen that

$$\lim_{n \to \infty} T_n(z)/\psi(z)^n = \frac{1}{2},$$

uniformly in compact subsets of $\mathbb{C} \backslash [-1, 1]$. Note that ψ maps $\mathbb{C} \backslash [-1, 1]$ conformally onto $\{w : |w| > 1\}$.

This type of asymptotic is called a *strong*, or *power*, or *Szegö* asymptotic. It implies the weaker *ratio* asymptotic

$$\lim_{n \to \infty} T_{n+1}(z)/T_n(z) = \psi(z)$$

and the still weaker *nth root* asymptotic (with a suitable choice of branches)

$$\lim_{n \to \infty} T_n(z)^{1/n} = \psi(z).$$

Asymptotically most of the analysis of orthogonal polynomials has involved one of these three asymptotics. It was Szegö [8] who proved in the early 1920s that if $\mu = \mu' \, dx$ is supported on $[-1, 1]$ and

$$(1) \qquad\qquad \int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1 - x^2}} \, dx > -\infty,$$

then there is a strong asymptotic of the form

$$\lim_{n \to \infty} p_n(\mu, z)/\psi(z)^n = G(z), \qquad z \notin [-1, 1],$$

where $G(z)$ is an "outer" function associated with $\log \mu'$. Conversely, the boundedness of $\{p_n(\mu; z)/\psi(z)^n\}_{n=0}^{\infty}$ is known to imply (1).

In 1940 Erdös and Turan [1] proved that if μ is supported on $[-1, 1]$ and μ' is positive a.e. in $[-1, 1]$, then there is an nth root asymptotic

$$\lim_{n \to \infty} p_n(\mu; z)^{1/n} = \psi(z), \qquad z \notin [-1, 1].$$

A little over 40 years later, Rahmanov [5] showed that the Erdös-Turan condition is sufficient for the stronger ratio asymptotic

$$\lim_{n \to \infty} p_{n+1}(\mu; z)/p_n(\mu; z) = \psi(z), \qquad z \notin [-1, 1].$$

This ratio asymptotic showed that such measures lie in *Nevai's class* $\mathscr{M}$, but the theory of that class is another story [4].

In the late 1960s and early 1970s Widom and Ullman gave more general criteria for nth root asymptotics, and, in spirit, the book of Stahl and Totik owes much to the ideas of Ullman.

But they have taken the theory much further. Their monograph is a very polished treatment of nth root asymptotics for orthonormal polynomials associated with general measures with compact support in $\mathbb{C}$. Perhaps there are still unsolved problems but surely accessible only to absolute experts with a very deep insight into orthogonal polynomials and potential theory. Every major theorem is accompanied by examples illustrating its sharpness, and many of their important results have not appeared elsewhere.

A fundamental ingredient in the formulation of their theorems and proofs is potential theory. This is natural, for the log of a polynomial is a potential with respect to a discrete measure; more precisely,

$$\log \left| \prod_{j=1}^{n} (z - z_j) \right| = - \int \log \left| \frac{1}{z - t} \right| \, d\nu(t),$$

where ν is the discrete measure of total mass n placing mass 1 at each zero z_j of the polynomial. It is this connection, and the power of weak convergence of sequences of measures, that has made potential theory an indispensable part of complex approximation and orthogonal polynomials in recent years. A forthcoming monograph of Saff and Totik will be devoted to these themes [6].

This has frightened off many would-be users/researchers, as the prospect of having to absorb an entire textbook such as Tsuji [9] or Landkof [3] or Hayman and Kennedy [2] has been too daunting. One of the major advantages of the book under review is a clear and detailed introduction to the relevant sections of potential theory, which begins in the opening chapters of this book and is completed in a lengthy appendix. For many who have no interest in orthogonal polynomials but need to apply potential theory to sequences of polynomials or to complex approximation, the appendix alone would justify buying the book.

Recall that if B is a Borel subset of $\overline{\mathbb{C}}$ with bounded complement, then the Green's function $g_B(z; \infty)$ for $\mathbb{C} \backslash B$ with pole at infinity is a function determined by the following three properties: (i) $g_B(z; \infty)$ is nonnegative and subharmonic in $\mathbb{C}$, and harmonic in the interior of B, except at ∞; (ii) $g_B(z; \infty) - \log|z|$ has a finite limit as $|z| \to \infty$; (iii) $g_B(z; \infty) = 0$ q.e. in $\mathbb{C} \backslash B$, that is, except possibly in a set of logarithmic capacity 0.

For a Borel measure μ with compact support, we define its *minimal-carrier Green function*

$$g_\mu(z; \infty) := \sup\{g_{\overline{\mathbb{C}} \backslash C}(z; \infty) : C \text{ is a bounded Borel set and } \mu(C) = \mu(\mathbb{C})\},$$

and its *minimal-carrier capacity* (here cap denotes logarithmic capacity)

$$c_\mu := \inf\{\operatorname{cap}(C) : C \text{ is a Borel set and } \mu(C) = \mu(\mathbb{C})\}.$$

(The sets C of full μ-measure are called *carriers* of μ.) Moreover, let Ω denote the unbounded component of $\mathbb{C} \backslash \operatorname{supp}[\mu]$. The authors show in Chapter 1 that

$$\limsup_{n \to \infty} |p_n(\mu; z)|^{1/n} \leq \exp(g_\mu(z; \infty))$$

uniformly in compact subsets of $\mathbb{C}$, and

$$\liminf_{n \to \infty} |p_n(\mu; z)|^{1/n} \geq \exp(g_\Omega(z; \infty))$$

locally uniformly in compact subsets of $\mathbb{C}\backslash \text{Co}(\text{supp}[\mu])$, where Co means convex hull. Moreover, the leading coefficients $\gamma_n(\mu)$ satisfy

$$1/\operatorname{cap}(\operatorname{supp}[\mu]) \le \liminf_{n\to\infty} \gamma_n(\mu)^{1/n} \le \limsup_{n\to\infty} \gamma_n(\mu)^{1/n} \le 1/c_\mu.$$

These results are mostly due to Ullman.

One interesting corollary is to arbitrary sequences of polynomials: If P_n is a polynomial of degree n that is not identically 0, $n \ge 1$, then uniformly in compact subsets of $\mathbb{C}$ we have

$$\limsup_{n\to\infty}(|P_n(z)|/\|P_n\|_{L_2(\mu)})^{1/n} \le \exp(g_\mu(z\,;\,\infty)).$$

The sharpness of the upper and lower bounds is carefully established.

In Chapter 2 the authors examine the connection between the asymptotic behaviour of $\gamma_n(\mu)^{1/n}$ and the zeros of $p_n(\mu\,;\,z)$. They begin by showing that all the zeros of $p_n(\mu\,;\,z)$ lie in $\text{Co}(\text{supp}[\mu])$, and in any compact set V of Ω, the number of zeros of $p_n(\mu\,;\,z)$ is bounded as $n \to \infty$. They then proceed to investigate the weak limits of the unit measures

$$\frac{1}{n}\nu_{p_n} := \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{jn}}, \qquad n \ge 1,$$

that place mass $1/n$ at each zero x_{jn}, $1 \le j \le n$, of $p_n(\mu\,;\,z)$. A rather special case of the results is that if $\text{supp}[\mu]$ is an interval and $c_\mu > 0$, then the assertion

$$(2) \qquad\qquad \lim_{n\to\infty} \gamma_n(\mu)^{1/n} = 1/\operatorname{cap}(\operatorname{supp}[\mu])$$

is equivalent to

$\frac{1}{n}\nu_{p_n}$ converges weakly to the equilibrium distribution of $\text{supp}[\mu]$, $n \to \infty$.

In Chapter 3 the notion of "regular asymptotic behaviour" is discussed. This is based on the equivalence of (2) to each of the following:

$$\lim_{n\to\infty} |p_n(\mu\,;\,z)|^{1/n} = \exp(g_\Omega(z\,;\,\infty))$$

locally uniformly in $\overline{\mathbb{C}}\backslash \text{Co}(\text{supp}[\mu])$;

$$\limsup_{n\to\infty} |p_n(\mu\,;\,z)|^{1/n} = 1 \quad \text{q.e. on the boundary of } \Omega.$$

If any of these hold, the measure μ is said to belong to Reg and to be regular. Several further characterisations of Reg are given, and the notion of regularity in $L_p(\mu)$ spaces is discussed—one of the main results being that regularity for one p implies regularity for all p. Chapter 4 deals with criteria for regularity, starting with the Erdös-Turan criterion, suitably generalized, Widom's and Ullman's criterion, and ending with several new criteria that are easily applicable.

The most original part of the monograph is Chapter 5, which deals with "localisation": If $\mu_K := \mu_{|K}$ is the restriction of μ to some compact set K, which is well behaved in the sense that it is regular, then what can we say about the *original* orthogonal polynomials $\{p_n(\mu\,;\,z)\}_{n=1}^{\infty}$? Moreover, if the outer boundary $\partial\Omega$ of $\text{supp}[\mu]$ can be decomposed into compact sets K_j such that μ_{K_j} is regular, can we deduce that μ is regular? These questions are carefully investigated, and even localization at a single point is discussed.

The reader unfamiliar with applications might question the utility of such a thorough study of nth root asymptotics—until he comes to Chapter 6. There the authors study rational interpolation, Padé approximation, and best rational approximation to Markov functions

$$f(z) = \int \frac{d\mu(x)}{x - z}.$$

In fact, much of the impetus for development of nth root asymptotics came from investigation of the rate of convergence of interpolation of functions analytic in some region, dating back at least to Walsh's monograph [10] and before.

One of the most elegant results of Chapter 6 is a powerful extension of work of the Gonchar school (based at the Steklov Institute in Moscow): Let V be a compact subset of $\overline{\mathbb{C}} \setminus \text{supp}[\mu]$ that is symmetric with respect to the real line, that is, $\overline{V} = V$. Assume moreover that $\text{cap}(V) > 0$, and let $\text{cap}(V, \text{supp}[\mu])$ denote the condenser capacity of the pair $(V, \text{supp}[\mu])$. Let $E_{nn}(f; V)$ be the error in best uniform rational approximation of f on V by rational functions of type (n, n) (this means numerator and denominator degree at most n) so that

$$E_{nn}(f; V) = \inf\{\|f - r\|_{L_\infty(V)} : r \text{ is a rational function of type } (n, n)\}.$$

Then

$$\lim_{n \to \infty} E_{nn}(f; V)^{1/(2n)} = \exp(-1/\text{cap}(V, \text{supp}[\mu]))$$

if and only if $\mu \in \text{Reg}$. Even when $\mu \notin \text{Reg}$, we have $\limsup \leq$.

The other applications in Chapter 6 are also of great interest, in particular, best $L^2(\mu)$ approximation of analytic functions, weighted approximations, Padé approximation of Markov functions by rational functions of type $(\lambda n, n)$ where λ is fixed, and determining sets.

Historical notes and comments are presented after the appendix. All in all, this is a well-written, well laid out, interesting research monograph, essential to anyone involved in complex approximation, orthogonal polynomials, rational approximation, and applications of potential theory in the plane.

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Combinatorial homotopy and 4-dimensional complexes, by Hans Joachim Baues. Walter de Gruyter, Berlin, 1991, 379 pp., $89.50. ISBN 0-89925-697-x.

Before discussing the material in the book under review, let me try to describe its mathematical context, starting with the notions of topological space and homeomorphism. Some technical language will be used without definitions. At their first appearance, such terms will be placed in quotation marks, in the hope that the context together with the reader's background will convey the general idea.

Ideally, one wants a classification theorem, or at least a characterization of topological type for significantly occurring spaces, in terms of a short list of invariants. As a general goal, this is out of reach. A breakthrough idea appeared in the work of J. W. Alexander, in his study of the topological invariance of simplicial homology. He showed that spaces of the same homotopy type have isomorphic homologies. Let us review the definitions. Two continuous maps $f, g : X \to Y$ are *homotopic* if there is a continuous map $H : X \times [0, 1] \to Y$ extending f and g on the subspaces $X \times \{0\}$ and $X \times \{1\}$ respectively. Two spaces X, Y have the same *homotopy type* provided there are maps $f : X \to Y$ and $g : Y \to X$ which are inverses up to homotopy; gf is homotopic to the identity map of X, and likewise for fg.

Often, but not always, computable topological invariants are actually invariants of homotopy type. Furthermore, the possibility of a short list of homotopy invariants characterizing significantly occurring homotopy types seems more within reach than the same problem for topological types. Even so, there are still major difficulties.

Some understanding of the problems for characterizing general homotopy types can be gained when one realizes that the basic algebraic tools in homotopy theory are often blind to local point set subtleties. For example, the Warsaw Circle (definition below) and a point, while not of the same homotopy type, look alike in the sense that they are "weakly equivalent". Roughly, this means that the inclusion of a point into the Warsaw Circle induces an isomorphism of any of the usual algebraic invariants, homotopy, homology, etc. Recall that the Warsaw Circle consists of the subspace of the Cartesian plane consisting of the graph of $y = \sin \frac{1}{x}$, $0 < x \le 1$, the vertical interval $\{0\} \times [-1, 1]$, and an arc from $(0, 0)$ to $(1, \sin 1)$ which misses the rest of the graph and the interval.