There are helpful notes at the ends of the chapters and a very complete bibliography at the end of the book. However, this reviewer noted one omission which may have been deliberate, since the book by Olds is quite elementary (*Continued fractions*, by C. D. Olds, no. 9 in the New Mathematical Library of the MAA). Elementary or not, this book contains some unusual items, such as the continued fraction expansion of π to 23 terms. This expansion does not seem to have any regularity, as contrasted with e, where we have $e - 1 = [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$. Rockett and Szüsz prove this nontrivial result in Chapter 1, whereas Olds merely states it.

The book presumes no knowledge of continued fractions, but a familiarity with the basic notions of number theory is taken for granted along with a "rudimentary acquaintance with Lebesgue theory". The probability theory needed is developed in the book. Although the authors include many remarks on the sources of their material, they refer the reader to other sources, such as the well-known book of André Weil, for the historical context of the classical results in the subject. There are no problems or exercises in the book.

In this review we have not attempted to give a catalog of all the topics covered in this book. For example, the authors present fairly recent or recent work by John Brillhart, Thomas Cusick, Harold Davenport, Mary E. Flahive, W. J. LeVeque, Alexander Ostrowski, and Wolfgang M. Schmidt, among others. Whether the book will take its place alongside the famous classics by A. Ya. Khintchine and Oscar Perron remains to be seen. Nevertheless, it provides a very substantial summary of the principal results in the theory of simple continued fractions. The authors have attempted to give the best possible results now known, with proofs that are the simplest and most direct. The book is an outstanding addition to the literature of mathematics.

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Transformations, transmutations, and kernel functions, Vol. 1, by Heinrich Begehr and Robert Gilbert. Longman Publishers, Harlow, 1992, 399 pp., $79.00. ISBN 3-582-02695-4

There has been an enormous amount of activity in the last 45 years or so on topics involving elliptic systems of differential operators, integral operators in the theory of elliptic partial differential equations (PDE), pseudoanalytic and hyperanalytic function theory, Riemann-Hilbert problems, etc. This was partly connected with the explosion of mathematical and applied mathematical activity after the Second World War but has its roots still further back in classical problems of elasticity and fluid dynamics, for example. We cite here for background a few books [1, 3–8, 14–18, 20, 22–31] that have been particularly important in chronicling and delimiting the directions of concern in the book under review. No attempt at completeness is intended here; many important

papers by various authors are also not mentioned, and we apologize for such omissions. We remark here also that in Volume 2 of the present work some evolution equations are considered in Chapter 8 as well as a sketch of Clifford analysis in Chapter 9; however, we will not dwell on these matters here. The other chapters in Volume 2 are Chapter 6, "Systems of Elliptic Equations", and Chapter 7, "Singularities of Solutions to Elliptic Equations".

One of the Leitmotive for the application of function-theoretic ideas in elliptic PDE is, of course, the possibility of exploiting various kernel functions and integral formulas that arise naturally from ellipticity. Historically such theories (kernel functions, pseudoanalytic functions, etc.) arose simultaneously as their need increased in applied problems. At the same time in the 1950s, for example, the theory of functional analytic methods in PDE was expanding rapidly (distributions, weak solutions, Sobolev spaces, etc.). There was and is, of course, some interaction, but the methods are often very different, and the choices of function spaces are best dictated by the methods. The function-theoretic methods described in this book usually retain the feeling of classical PDE (i.e., calculus!), while the functional analytic approach often looks more like operator theory, with a corresponding lack of geometric perspective at times. One of the nice features of the book under review is frequently to combine function-theoretic methods and functional analytic techniques in various places to make the best use of both worlds and exhibit how a productive interaction can be achieved.

Let us indicate a few of the typical questions, problems, and methods involved in function-theoretic methods for PDE. We do not try to survey the whole area (it is much too vast) but rather pick topics directly from the book. Thus I will indicate chapter contents in a few sentences, following the book's preface, and then illustrate the problems and techniques with very simple examples and formulas to provide a background idea of what is going on. In fact, it is stated in the book's preface that it explicitly limits its scope to serve as an "embellishment" of [26] together with numerous applications, and we will correspondingly limit the scope of this review. Any survey of all problems and all techniques and all applications involving function-theoretic methods would itself be of book length, if not impossible, and would probably be useless in its intricacy. The chapter headings are: (1) "Analytic Methods in the Theory of Elasticity", (2) "Boundary Value Problems", (3) "The Schwartz Function and Hele-Shaw Flows", (4) "Transmutations", and (5) "Elliptic PDE in Two Variables with Analytic Coefficients"; the contents of Volume 2 are indicated above. The main themes are the construction of solutions to (usually) elliptic differential equations and/or systems using analytic methods. This allows one to consider "short series" approximations of solutions (via several terms in a generating or reproducing kernel or via asymptotic series) in contrast to numerical "long series" approximations. There is a certain amount of material in such computational directions on which I will not comment due to lack of expertise; suffice it to say that it seems like a good idea and appears to be well done.

Going to Chapter 1, the book's description indicates here a description of some of the classical applications of function theory to plane elasticity (in particular, elastic shell theory). Specifically one looks at certain typical problems from elasticity leading to equations

$$(1) \qquad\qquad \overline{\partial}_z U + AU + B\overline{U} = F$$

where $U = u(x, y) + iv(x, y)$ and $\overline{\partial}_z = \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. A sketch of the Vekua method is given where solutions to the homogeneous problem $(F = 0)$ are called generalized analytic functions. One works in a "suitable" region G, and we also omit specific hypotheses on coefficients, etc. The details in the book are admirably explicit. One describes the Cauchy-Pompeiu operator $(\zeta = \xi + i\eta)$

$$(2) \qquad T_G f = -(1/\pi) \int_G (f(\zeta)/(\zeta - z)) \, d\xi \, d\eta$$

and its role in solving equations (1) (note $\overline{\partial}_z w = f \sim w = \phi + T_G f$ where ϕ is analytic). One describes also some basic generalized hyperanalytic function theory and discusses hypermembranes (cf. [28]). A general Cauchy-Pompeiu representation is indicated, and one solves equations of the form

$$(3) \qquad \overline{D}w = \overline{\partial}_z w + q_1 \partial_z w + q_2 \overline{\partial_z w} = 0 \quad \text{or} \quad \overline{D}w + aw + b\overline{w} = f$$

where the q_i are hyperanalytic and nilpotent and a, b, f are hypercomplex. Here generalized hyperanalytic functions go back to the hyperanalytic functions of A. Douglis where $U_x + AU_y = 0$ (A a $2n \times 2n$ matrix with no real eigenvalues). After complexification one reduces matters to equations $w_x + Jw_y = 0$ where J is an $n \times n$ Jordan block $J = \lambda I + e$ ($e \sim$ subdiagonal 1). The algebra of hypercomplex elements is generated by $\{1, i, e\}$, and one considers equations, e.g.,

$$(4) \quad Dw = \overline{\partial}_\zeta w + Q\partial_\zeta w = 0; \qquad Q = \sum_1^{n-1} q_k e^k, \quad \text{or} \quad Dw + aw + b\overline{w} = 0$$

for suitable hypercomplex Q, a, b. Solutions of $Dw = 0$ are classical hyperanalytic functions, and one has operators

$$(5) \qquad J_G f = -(1/\pi) \int_G t_\zeta(\zeta)(f(\zeta)/(t(\zeta) - t(z))) \, d\xi \, d\eta \quad \text{[Pompeiu]},$$

$$C_G f = (1/2\pi i) \int_{\partial G} (f(\zeta)/(t(\zeta) - t(z))) \, d\zeta \quad \text{[Cauchy]}$$

leading to $w = C_G w + J_G g$ as a solution to $Dw = g$. Here $t(z)$ is a so-called "generating solution" of $Dw = 0$. There is a function theory for hyperanalytic functions with integral formulas and kernels extending (5); and in studying the analogous boundary value problems (BVP), one deals with Riemann-Hilbert (RH) problems for hyperanalytic functions. One says that if $C = G^+ \cup \Gamma \cup G^-$ with Γ a system of $m + 1$ smooth curves, the Hilbert problem (H) is to find a piecewise hyperanalytic function w in C (e.g., $DW = aw + b\overline{w}$, $w(\infty) = 0$), continuous in G^+ and G^-, and satisfying on Γ the jump condition $w^+ - Hw^- = h$. Here $w^\pm$ are limiting values from $G^\pm$, and h, H are hypercomplex valued. Similarly, a Riemann problem (R) in the hypercomplex case involves finding a solution to $DW = aw + b\overline{w}$ in G with $\text{Re}(\gamma w) = \phi$ on Γ. Such RH problems (terminology differs here) are of enormous importance in many areas of mathematical physics (see, e.g., [13, 21, 12] for problems in soliton theory), and for hyperanalytic functions one can refer to [28]. There are hyperanalytic Plemelj-Sokhotskij formulas, and a number of applications of RH problems occur in Begehr and Gilbert's book.

For Chapter 2 the preface indicates the solution of the RH problem for the hypermembrane in §1. Then two sections are devoted to kernel function methods. Section 4 is on the seminal method of layers due to Fichera, and §5 deals with T-complete families d'après Herrera. Numerous examples are given along with a Macsyma program for T-complete families. Finally, a survey of recent results for boundary value problems for nonlinear first-order elliptic systems is given. To be more specific, we recall from classical arguments based on Green's theorems for $\Delta u = Pu$ ($P \geq 0$, say) in $D \subset R^n$ that one can define a reproducing kernel

$$(6) \qquad K(x, y) = N(x, y) - G(x, y) = \sum_{1}^{\infty} \phi_m(x)\phi_m(y),$$

$$u(x) = -\int_{\partial G} u(y)\partial_\nu K(y, x)\, d\sigma_y,$$

$$u(x) = -\int_{\partial G} (K(y, x)\partial_\nu u(y)\, d\sigma_y$$

(real functions), and this has a matrix version as well. For $n \times n$ systems $\Delta U - C(x, y)U = 0$, C positive definite Hermitian, say, one can find $K(x, y) = \sum_{1}^{\infty} U_k(x)U_k^*(y)$. These represent basic Bergman kernels. One has various beautiful integral equations and operator-theoretic properties connected to all this, some of which are used in the book. Extensions to higher-order operators $L(u) = \Delta^p u + (-1)^p Qu$ and to polyanalytic functions u satisfying $\bar{\partial}_z^n u = 0$ are also possible. For background here see, for example, [2, 3, 5, 27, 28, 41] and note eventual connections to Bochner-Martinelli kernels and a whole package of beautiful theory in several complex variables. Sections 4 and 5 provide a nice collection of results on rigorous applied mathematics, and §6 is a valuable survey. One notes that the authors have an excellent grasp of what not to say in writing a book. Enough detail is provided, general ideas are clearly sketched, background theorems are stated with full definitions of terms and references, and adequate proofs are given when appropriate.

From the preface Chapter 3 introduces the very important Schwartz function and gives generalizations to higher dimensions. Connections to Hele-Shaw moving boundary value problems are stressed, and a fairly thorough investigation of the Hele-Shaw problem (viscous flow between 2 slightly separated plates with injection of fluid) and its generalization to anisotropic flows and higher dimensions are given. The presentation is developed primarily along the ideas of Gustafsson, Elliot, and Janovski, and their generalizations by Begehr, Gilbert, and Shi. A new method due to Gilbert and Wen is also suggested for solving the anisotropic Hele-Shaw problem. I found all this quite fascinating. Connections to the "measure-theoretic" work of Sakai are also indicated. The n-dimensional generalizations (which can be used to approximate the propagation of a liquid front through a porous medium) can be phrased in various ways (e.g., as obstacle problems with variational inequalities), and a number of interesting results are given. The whole chapter is one of the best examples I have seen of the interaction of function theory, functional analysis, and hard analysis in solving problems in PDE, but there are too many details to summarize much here.

Chapter 4 begins with a development of transmutation theory d'après [10, 11] and presents the method of ascent (i.e., the Bergman-Gilbert operator) as

a transmutation. This is then applied to problems in underwater acoustics. *T*-complete families plus transmutation ideas are used to solve BVP, and in particular it is shown how the farfield for a stratified ocean of finite depth can be computed. The last section presents recent work of Begehr and Hile on entire solutions of Helmholz-type equations. In more detail we remark that transmutation operators (or transformation operators) go back to Gelfand, Levitan, Naimark, Marchenko, Lions, Delsarte,... (cf. [32–35]) and were organized and further developed in certain directions by the reviewer in [10, 11] (cf. also [9, 12–14] and the book being reviewed for reference to work of Bragg, Dettman, Colton, Donaldson,...). Transmutation here basically involves intertwining two linear differential operators P and Q via $PB = BQ$, acting on suitable objects f, such that, for example, the spectral theory of a simple operator P can be transported to that of Q via B. In particular, the direction emphasized in [9–13] is that a transmutation kernel for B may often be written as a spectral pairing $B(x, y) = \langle \phi(x, \lambda), \psi(y, \lambda) \rangle_\Lambda$ where ϕ (resp. ψ) is a generalized eigenfunction of Q (resp. P) and Λ is a suitable measure. For example, if for suitable $\sigma \int \psi(y, \lambda) \psi(y, \mu) d\sigma(y) = \delta_\Lambda(\lambda - \mu)$ (where $\langle f(\lambda), \delta_\Lambda(\lambda - \mu) \rangle_\Lambda = f(\mu)$), then formally

$$(7) \quad B\psi = \int B(x, y) \psi(y, \mu) d\sigma(y) \sim \langle \phi(x, \lambda), \int \psi(y, \lambda) \psi(y, \mu) d\sigma(y) \rangle_\Lambda$$

$$= \langle \phi(x, \lambda), \delta_\Lambda(\lambda - \mu) \rangle_\Lambda = \phi(x, \mu).$$

This transport of generalized eigenfunctions can be exploited in many ways, especially in inverse scattering theory (cf. [10–13, 33, 35]). In particular, a certain important operator arises in work of Bergman and Gilbert which can be expressed transmutationally and given a spectral form. Thus, for example, solutions of $\Delta u + F(r^2)u = 0$ in R^2 can be represented in the form

$$(8) \qquad u(x) = \int_{-1}^1 E(r^2, t) H(x(1 - t^2)) \, dt / (1 - t^2)^{1/2}$$

$$= h(x) + \int_0^1 \sigma^{n-1} G(r, 1 - \sigma^2) h(x\sigma^2) \, d\sigma$$

$(n = 2)$ where $h = \int_{-1}^1 H(x(1 - t^2) \, dt / (1 - t^2)^{1/2}$ with $\Delta H = \Delta h = 0$. The E and G refer to Bergman type kernels, and Gilbert showed (cf. [26]) that the G formula extends to R^n (as indicated) with $G(r, 1 - \sigma^2) = -2z\partial_3 R(z, \bar{z}, z\sigma^2, 0)$ independent of n where R is a complex Riemann function. This is now referred to as the Bergman-Gilbert operator. Incidentally, the use of Riemann functions and Goursat problems occurs quite frequently in the kernel theory. In §2 a number of innovative developments of transmutation ideas and integral operator techniques are applied to direct and inverse problems of ocean acoustics, near and farfield asymptotics, and so on.

Finally, Chapter 5 is devoted to a detailed treatment of the classical Bergman-Vekua theory in the plane. Many extensions and special cases are also included. In particular, Gilbert's method of ascent and extensions is developed. The chapter is nicely written, and I have little to say beyond the remark that much of A. Weinstein's generalized axially symmetric potential theory can be developed via transmutation, as in [10].

Overall this is a very valuable, well-written book with a nice balance between theory and applications. It illustrates very well what rigorous applied mathematics can accomplish and gives a good background for the techniques used. It also contains over 500 references to the literature. Both authors are authorities in the subject matter and are reporting on the state of the art. There is a very productive infusion of classical and modern material from East and West Europe, the former Soviet Union, China, and North America.

References

1. K. Bauer and S. Ruscheweyh, *Differential operators for PDE and function theoretic applications*, Springer, New York, 1980.

2. S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, FL, 1992.

3. S. Bergman, *The kernel function and conformal mapping*, Amer. Math. Soc., Providence, RI, 1950.

4. _____, *Integral operators in the theory of linear PDE*, Ergeb. Math. Grenzgeb. (3), vol. 23, Springer, Berlin, 1961.

5. S. Bergman and M. Schiffer, *Kernel functions and elliptic PDE in mathematical physics*, Academic Press, New York 1953.

6. L. Bers, *Theory of pseudoanalytic functions*, NYU Lecture Notes, 1953.

7. _____, *Mathematical aspects of subsonic and transonic gas dynamics*, Wiley-Interscience, New York, 1958.

8. A. Bitsadze, *Boundary value problems*, North-Holland, Amsterdam, 1968.

9. R. Carroll, *Transmutation and operator differential equations*, North-Holland, Amsterdam, 1979.

10. _____, *Transmutation, scattering theory, and special functions*, North-Holland, Amsterdam, 1982.

11. _____, *Transmutation theory and applications*, North-Holland, Amsterdam, 1985.

12. _____, *Mathematical physics*, North-Holland, Amsterdam, 1988.

13. _____, *Topics in soliton theory*, North-Holland, Amsterdam, 1991.

14. R. Carroll and R. Showalter, *Singular and degenerate Cauchy problems*, Academic Press, New York, 1976.

15. D. Colton, *PDE in the complex domain*, Pitman, London, 1976.

16. _____, *Analytic theory of PDE*, Pitman, London, 1980.

17. D. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley, New York, 1983.

18. R. Courant and D. Hilbert, *Methods of mathematical physics*, Wiley-Interscience, New York, 1962.

19. A. Dzhuraev, *The method of singular integral equations*, "Nauka", Moscow, 1987.

20. _____, *Degenerate and other problems*, Longman, New York, 1992.

21. L. Faddeev and L. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, New York, 1987.

22. G. Fichera, *Linear elliptic differential systems and eigenvalue problems*, Lecture Notes in Math., vol. 8, Springer-Verlag, New York, 1965.

23. A. Friedman, *Variational principles and free BVP*, Wiley, New York, 1982.

24. F. Gakhov, *Boundary value problems*, Permagon Press, Oxford, 1966.

25. P. Garabedian, *PDE*, Wiley, New York, 1964.

26. R. Gilbert, *Constructive methods for elliptic equations*, Lecture Notes in Math., vol. 365, Springer-Verlag, New York, 1974.

27. _____, *Functions theoretic methods in PDE*, Academic Press, New York, 1969.

28. R. Gilbert and J. Buchanan, *First order elliptic systems: A function theoretic approach*, Academic Press, New York, 1983.

29. W. Haack and W. Wendland, *Lectures on Pfaffian and partial differential equations*, Permagon Press, Oxford, 1972.

30. M. Kracht and E. Kreyszig, *Methods of complex analysis in PDE with applications*, Wiley-Interscience, New York, 1988.

31. V. Kupradze, *Potential methods in the theory of elasticity*, Israel Prog. Scient. Transl., Jerusalem, 1965.

32. B. Levitan, *Theory of generalized shift operators*, 2nd ed, "Nauka", Moscow, 1973.

33. _____, *Inverse Sturm-Liouville problems*, VNU Science Press, Utrecth, 1987.

34. J. L. Lions, *Equations différentielles opérationnelles*, Springer Berlin, 1961.

35. V. Marchenko, *Sturm-Liouville operators and their applications*, Kiev, 1977.

36. E. Meister, *Randwertaufgaben der Funktionentheorie*, Teubner, Stuttgart, 1983.

37. C. Miranda, *PDE of elliptic type*, Springer, Berlin, 1970.

38. N. Muskhelishvili, *Singular integral equations*, Noordhoff, Groningen, 1953.

39. _____, *Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1953.

40. I. Privalov, *Randeigenschaften analytischen Funktionen*, VEB Deutsch. Verlag Wissen., Berlin, 1956.

41. R. Range, *Holomorphic functions and integral representations in several complex variables*, Springer, New York 1986.

42. Y. Rodin, *The Riemann BVP on Riemann surfaces*, Reidel, Dordrecht, 1988.

43. M. Schiffer and D. Spencer, *Functionals of the finite Riemann surface*, Princeton Univ. Press, Princeton, NJ, 1954.

44. S. Sitnik and V. Katrakhov, *Analytical theory of transmutations with applications*, Pitman Press, London (to appear).

45. W. Tutschke, *Partielle komplexe Differentialgleichungen in einer und in mehreren komplexen Variablen*, VEB Deutsch. Verlag Wissen., Berlin, 1977.

46. I. Vekua, *Generalized analytic functions*, Permagon Press, Oxford, 1962.

47. _____, *New methods for solving elliptic equations*, North-Holland, Amsterdam, 1967.

48. G. Wen and H. Begehr, *Boundary value problems for elliptic equations and systems*, Longman, Essex, 1990.

49. W. Wendland, *Elliptic systems in the plane*, Pitman, London, 1979.

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Classical sequences in Banach spaces, by Sylvie Guerre-Delabrière. Marcel Dekker, Inc., New York, 1992, xiv + 207 pp., $99.75. ISBN 0-8247-8723-4

For over sixty years a central question in modern analysis has been: Which Banach spaces contain *almost isometric copies* of one of the classical sequence spaces c_0 or l_p, for some $1 \leq p < \infty$? (A Banach space X contains *almost*