quotient space of usual function spaces, hence the necessity of a factorization of symbols, which is a difficult problem by itself. The corresponding results extend the classical work by M. Krein and many other authors. The real difficulty here is an instability of the factorization, so variable coefficients are not considered.

The strong point of the book is that it treats the convolution equations with an exhaustive thoroughness which was without doubt difficult to achieve. So I definitely recommend the book to the experts who work in this area. Most parts of it might also be useful for graduate students to supplement an advanced PDE course.

MIKHAIL A. SHUBIN
NORTHEASTERN UNIVERSITY
E-mail address: shubin@neu.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 1, January 1994

0273-0979/94 $1.00 + $.25 per page

Subgroups of Teichmüller modular groups, by Nikolai V. Ivanov. American Mathematical Society, Providence, RI, 1992, xii + 127 pp., $107.00. ISBN 0-8218-4594-2.

Let us consider a compact orientable surface S, possibly with boundary. The *Teichmüller modular space* Mod_S of S is the group of isotopy classes of orientation-preserving homeomorphisms of S. Namely, Mod_S is the quotient of the group of all orientation-preserving homeomorphisms of S under the equivalence relation which identifies two such homeomorphisms f_0, $f_1 : S \to S$ when they can be connected by a continuous family of homeomorphisms $f_t : S \to S$, $t \in [0, 1]$.

This ubiquitous group plays a fundamental role in the theory of Riemann surfaces and in algebraic geometry because of its connection to the moduli space of curves. To simplify the exposition, let us temporarily restrict attention to the case where S has no boundary. The *moduli space* $\mathcal{M}_S$ is the space of all complex curves with the same topological type as S, where we identify two such curves when there is a holomorphic homeomorphism between them. Another way to express this is to say that $\mathcal{M}_S$ is the space of all complex structures on S, where we identify two complex structures when there is a homeomorphism $f : S \to S$ sending one to the other. Although the moduli space is probably geometrically more relevant, a more convenient space is the *Teichmüller space* $\mathcal{T}_S$. The definition of $\mathcal{T}_S$ is similar to the second definition of the moduli space, but in this case we identify two complex structures on S only if there is a homeomorphism *isotopic to the identity* sending one to the other. One of the advantages of the Teichmüller space is that it can be given nice sets of coordinates and that it is a contractible manifold. There is an obvious action of Mod_S on $\mathcal{T}_S$, whose quotient is clearly $\mathcal{M}_S$; in addition, this action is properly discontinuous. In other words, the group Mod_S plays the same role in the theory of Riemann surfaces of higher genus as the classical modular group $\mathrm{SL}_2(\mathbb{Z})$ in the theory of elliptic curves.

If the action of Mod_S on $\mathscr{T}_S$ were free, the algebraic topology of the quotient $\mathscr{M}_S$ would be the same as that of the group Mod_S, since $\mathscr{T}_S$ is contractible. This is not the case, but anyway Mod_S has many finite index subgroups acting freely on $\mathscr{T}_S$. So, the algebraic topology of these finite index subgroups is the same as that of the corresponding finite branched covers of the moduli space. In other words, the Teichmüller modular group Mod_S and the moduli space $\mathscr{M}_S$ have the same virtual algebraic topologic properties. Another way to say pretty much the same thing is that Mod_S encaptions the *orbifold* algebraic topology of $\mathscr{M}_S$. In particular, Mod_S is the orbifold fundamental group of $\mathscr{M}_S$ [Th1, §13]. (The notion of orbifold is specially tailored to study quotients of manifolds by groups actions which are properly discontinuous but not necessarily free.)

We should also mention that a complex structure on S is equivalent to a conformal structure, namely, to a Riemannian metric defined modulo pointwise multiplication by functions $S \to \mathbb{R}$. This pushes the moduli space $\mathscr{M}_S$ and, consequently, Mod_S into the realm of differential geometry. For instance, a favorite trick when one tries to minimize a functional over the space of all metrics on S is to first minimize it in a given conformal class and then to vary this class over the moduli space. Another connection to differential geometry is Poincaré's observation that a conformal structure on S is realized by a unique metric of constant curvature, modulo scaling by a constant function (assuming S connected). Thus, the moduli space is also the projectivized space of all constant curvature metrics on S.

Topologists, and in particular low-dimensional topologists, encounter the group Mod_S in a different context. In fact, the context is so different that they usually call this Teichmüller modular group by another name, either the *mapping class group* of S or the *homeotopy group* of S. A typical construction in which this arises is the following: Let S be a transversely orientable surface in a 3-dimensional manifold M, and consider a homeomorphism $f: S \to S$. Then we can construct a new manifold M_f by splitting open M along the surface S and then gluing back together by f the two copies of S in the boundary of the split manifold. One easily shows that the homeomorphism type of M_f remains unchanged if we modify f by an isotopy, namely, that up to homeomorphism the manifold M_f depends only on the class of f in Mod_S.

Finally, there is a special case of interest when the surface S is *planar,* namely, when it is homeomorphic to the complement of n disjoint open discs in the sphere S^2. Then Mod_S is basically Artin's group B_{n-1}, the *braid group* on $(n-1)$ strands. More precisely, Mod_S is in this case the spherical braid group on n strands and is the quotient of B_{n-1} by its (infinite cyclic) center; see for instance [Bir]. Still in this case Mod_S is also the fundamental group of the space of all configurations of n disjoint points on the sphere S^2.

Although Artin quickly found a finite presentation for the braid group and provided a relatively good understanding of its algebraic structure, the algebraic structure of the general Teichmüller modular group has been more elusive. Dehn proved in [De] that Mod_S is finitely generated, a fact that was apparently relatively forgotten until the 1960s. But it was only in the mid-1970s that Mod_S was proved to be finitely presented by J. McCool and, more geometrically, by

A. Hatcher and W. P. Thurston. By comparison, J. Nielsen gave a finite presentation of the strongly related automorphism group of the free group as early as 1924.

There are, however, two algebraic questions of primary importance to topologists—namely, the word problem and the conjugacy problem for Mod_S, which have received relatively satisfactory answers. The word problem, namely, the problem of deciding whether two products of generators of Mod_S represent the same element in this group, was basically solved by Dehn when he solved the word problem and the conjugacy problem for the fundamental group of S. In the late 1970s, G. Hemion provided a combinatorial algorithm which solves the conjugacy problem for Mod_S; namely, it enables one in principle to decide whether or not two elements of Mod_S are conjugate. This followed a more effective solution for the conjugacy problem in the braid group given by F. A. Garside in the late 1960s. Garside's original algorithm has since been improved by Thurston to be computationally more efficient.

However, more or less at the same time as Hemion's work, the conjugacy problem in Mod_S received a sudden boost with the work of Thurston on the dynamics of surface homeomorphisms [Th2, FLP]. Thurston proved that, given an element of Mod_S, there is a family $\mathscr{C}$ of disjoint simple closed curves in S (possibly empty) and there is a homeomorphism $f : S \to S$ representing this element of Mod_S such that:

(i) none of the curves of $\mathscr{C}$ can be deformed to a point or to a boundary component;

(ii) $f(\mathscr{C}) = \mathscr{C}$;

(iii) if $S_{\mathscr{C}}$ is the surface obtained by cutting S open along $\mathscr{C}$ and if $f_{\mathscr{C}}$ is the homeomorphism of $S_{\mathscr{C}}$ induced by f, then $f_{\mathscr{C}}$ is isotopic to a homeomorphism $g : S_{\mathscr{C}} \to S_{\mathscr{C}}$ such that, for each component S_0 of $S_{\mathscr{C}}$, the restriction $g_{|S_0}$ is either periodic or pseudo-Anosov.

In (iii) *periodic* means, of course, that there is an n such that $g^n_{|S_0}$ is the identity.

Although the precise definition is a little too technical for the sake of this review, we can say that *pseudo-Anosov homeomorphisms* are surface homeomorphisms with nice symbolic dynamic properties. As an illustration we can extract from the dynamical properties of pseudo-Anosov homeomorphisms the following remarkable homotopic property. Endow S with any Riemannian metric and, for every closed curve γ in S, let the *minimum length* $l(\gamma)$ of γ be the minimum of the lengths of all curves in S obtained by deformation of γ. Then, if $g_{|S_0}$ is pseudo-Anosov, there is a number $\lambda > 1$ associated to $g_{|S_0}$ such that, for every curve γ in S_0 that cannot be deformed to a point or a boundary component, the minimum length $l(f^n(\gamma)) = l(g^n(\gamma))$ grows exponentially like λ^n as n tends to ∞.

If, in addition, we require that $\mathscr{C}$ is minimal for the above properties (i) to (iii), namely, that no proper subfamily of $\mathscr{C}$ satisfies these properties for some f representing the element of Mod_S considered, then it turns out that the family $\mathscr{C}$ is unique up to isotopy and that the class of $f_{\mathscr{C}}$ in $\mathrm{Mod}_{S_{\mathscr{C}}}$ is uniquely determined. We therefore have a canonical decomposition of the isotopy class $[f] \in \mathrm{Mod}_S$ into pieces which are either periodic or pseudo-Anosov.

This canonical decomposition had already been identified by Nielsen in the

1930s (although he did not discover the pseudo-Anosov phenomenon). It is called the *Nielsen-Thurston* decomposition of $[f] \in \mathrm{Mod}_S$.

It turns out that there is a relatively efficient algorithm, developed by L. Mosher, to solve the conjugacy problem for the periodic and pseudo-Anosov pieces, so this Thurston-Nielsen decomposition leads to another algorithm to solve the conjugacy problem in Mod_S. But it is also very important from a theoretical point of view.

Somewhere in the book under review Ivanov makes the judicious observation that in many senses the Nielsen-Thurston decomposition can be viewed as an analog of the Jordan decomposition theorem for matrices.

This leads us to the major theme of this monograph.

As indicated earlier, the Teichmüller modular group Mod_S is a higher genus generalization of the classical modular group $\mathrm{SL}_2(\mathbb{Z})$. But there are other higher-dimensional generalizations of $\mathrm{SL}_2(\mathbb{Z})$, namely, $\mathrm{SL}_n(\mathbb{Z})$ or the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ and, more generally, arithmetic linear groups and linear groups. It turns out that Mod_S shares many properties with arithmetic groups.

For instance, J. Harer and N. V. Ivanov have shown that the cohomologies of Mod_S and of arithmetic groups have many common features. The article [Iv1] is a very good reference for this.

When S is "small"—namely, when each of its component is a torus, a torus minus a disk, or a planar surface with at most 4 boundary components—there are good reasons for the analogy, because Mod_S is basically the trivial group or $\mathrm{SL}_2(\mathbb{Z})$, modulo finite index extensions or quotients. However, we should mention that Mod_S is otherwise never isomorphic to an arithmetic group and this for relatively elementary reasons [Iv2].

It is still a very open problem to know if Mod_S can be isomorphic to a linear group, namely, can be embedded in some $\mathrm{GL}_n(\mathbb{R})$. This is not even known for the braid group. For over fifty years the Burau representation had provided a reasonable candidate for an embedding of the braid group into some $\mathrm{GL}_n(\mathbb{R})$, but these hopes were recently proved to be unfounded by J. A. Moody. There is actually now a strong suspicion that Mod_S might not be linear after all. Indeed, E. Formanek and C. Procesi very recently proved that the automorphism group of the free group, which shares numerous features with Mod_S, is not linear.

The monograph is devoted to proving that Mod_S satisfies many of the classical group-theoretic properties of linear groups. Among these properties we can mention:

(a) the Tits alternative theorem, which says that a linear group either contains a solvable subgroup of finite index or contains a free group on two generators;

(b) the Margulis-Soifer theorem, which says that a finitely generated linear group contains either a solvable group of finite index or uncountably many maximal subgroups of infinite index;

(c) the Platonov theorem, which says that the intersection of the maximal subgroups of a finitely generated linear group is nilpotent;

(d) the residual finiteness of every finitely generated linear group.

Most of these results were first proved by the author, although similar lines of ideas had also been independently investigated by J. S. Birman, A. Lubotzky, and J. McCarthy.

These results come in support of the conjecture that Mod_S is linear, but they are of interest by themselves. What is also quite interesting here is the way these properties of Mod_S are obtained. Indeed, the proof of these purely group-theoretic properties heavily relies on the machinery of the Nielsen-Thurston decomposition and on the dynamical properties of pseudo-Anosov homeomorphisms. More precisely, Thurston introduced a canonical compactification of Teichmüller space, whose points of infinity correspond to projective measured laminations, which basically are diffused simple closed curves on the surface S. The action of Mod_S on Teichmüller space continuously extends to an action on this compactification. The basic idea behind Ivanov's arguments is a careful analysis of the dynamics of the action of subgroups of Mod_S on Thurston's compactification of Teichmüller space.

The linearity question for Mod_S was clearly the original motivation for these results and is of prime philosophical importance. However, as the author points out, it is of little technical relevance to these properties of Mod_S. Indeed, even if the linearity problem was settled, the proofs presented by the author would be much more direct and carry much more geometric information than the use of their linear counterparts. The author makes an even deeper observation. He notices that his proofs actually show that not only is Mod_S similar to a linear group but the action of Mod_S on Thurston's space of measured laminations is similar to a linear action. Since this action is definitely not linear, this suggests that these properties are in no way specifically linear and are valid in a higher level of generality.

References

[Bir] J. S. Birman, *Braids, links and mapping class groups*, Ann. of Math. Stud., vol. 84, Princeton Univ. Press, Princeton, NJ, 1974.

[De] M. Dehn, *Die Gruppe der Abbildungsklassen*, Acta Math. **69** (1938), 135–206.

[FLP] A. Fathi, F. Laudenbach, and V. Poenaru, *Travaux de Thurston sur les surfaces*, Astérisque, vol. 66-67, Soc. Math. France, Paris, 1979.

[Iv1] N. V. Ivanov, *Complexes of curves and the Teichmüller modular group*, Uspekhi Mat. Nauk **42** (1987), 49–91; English transl., Russian Math. Surveys **42** (1987), 55–107.

[Iv2] ———, *Teichmüller modular groups and arithmetic groups*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov **167** (1988), 95–110; English transl., J. Soviet Math. **52** (1990), 2809–2818.

[Th1] W. P. Thurston, *The topology and geometry of 3-manifolds*, Lecture notes, Princeton Univ., 1976–79.

[Th2] ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431.

Francis Bonahon
University of Southern California
E-mail address: fbonahon@math.usc.edu